

TIME-DEPENDENT ATTRACTOR FOR THE OSCILLON EQUATION

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ABSTRACT. We investigate the asymptotic behavior of the nonautonomous evolution problem generated by the Oscillon equation

$$\partial_{tt}u(x,t) + H\partial_tu(x,t) - e^{-2Ht}\partial_{xx}u(x,t) + V'(u(x,t)) = 0, \quad x \in (0,1), t \in \mathbb{R},$$

with periodic boundary conditions, where $H > 0$ is the Hubble constant and V is a nonlinear potential of arbitrary polynomial growth. After constructing a suitable dynamical framework to deal with the explicit time dependence of the energy of the solution, we establish the existence of a regular global attractor $\mathcal{A} = \mathcal{A}(t)$. The kernel sections $\mathcal{A}(t)$ have finite fractal dimension.

1. INTRODUCTION

The present paper is devoted to the study of the nonautonomous evolution problem generated by the equation

$$(1.1) \quad \partial_{tt}u(x,t) + H\partial_tu(x,t) - e^{-2Ht}\partial_{xx}u(x,t) + V'(u(x,t)) = 0, \quad x \in (0,1), t \in \mathbb{R},$$

where V is a nonlinear potential of polynomial growth satisfying natural dissipativity conditions. Equation (1.1) is the Klein-Gordon equation, with the given nonlinear potential, for a scalar field on a manifold with a Robertson-Walker metric corresponding to an expanding universe with Hubble constant $H > 0$, and is referred to here as the *oscillon* equation.

The physical motivation for the mathematical development in this paper stems from a role of Equation (1.1) in recent cosmological theories. It has been suggested that long-lived, localized, oscillating solutions to the equations of the standard model of particle physics may be useful in breaking thermodynamic equilibrium. Long-lived “oscillons” have been found to occur in a numerical simulation of the simplified model considered here, in one space dimension, for a nonlinear potential V of appropriate form [15], as shown in Fig. 1. Oscillons have also been studied in the context of other simple models of early universe phase transitions [1, 11, 18, 26]. These coherent structures seem to capture the essential features of the oscillon phenomenon in simulations of the full standard model in three-dimensional space. Localized oscillating solutions had been previously found in models of vibrating granular media [31]. In the cosmological context, the external forcing is replaced by thermalization of initial conditions, and the dissipation is scale-independent. The “friction” term, containing the first time-derivative of the field, results from the expansion.

The structures seen in Fig. 1a are localized, low-frequency oscillations. The equation is written in a coordinate system in which the expansion of the universe is not apparent. Rather, the oscillons, which are of constant physical width, steadily decrease in width

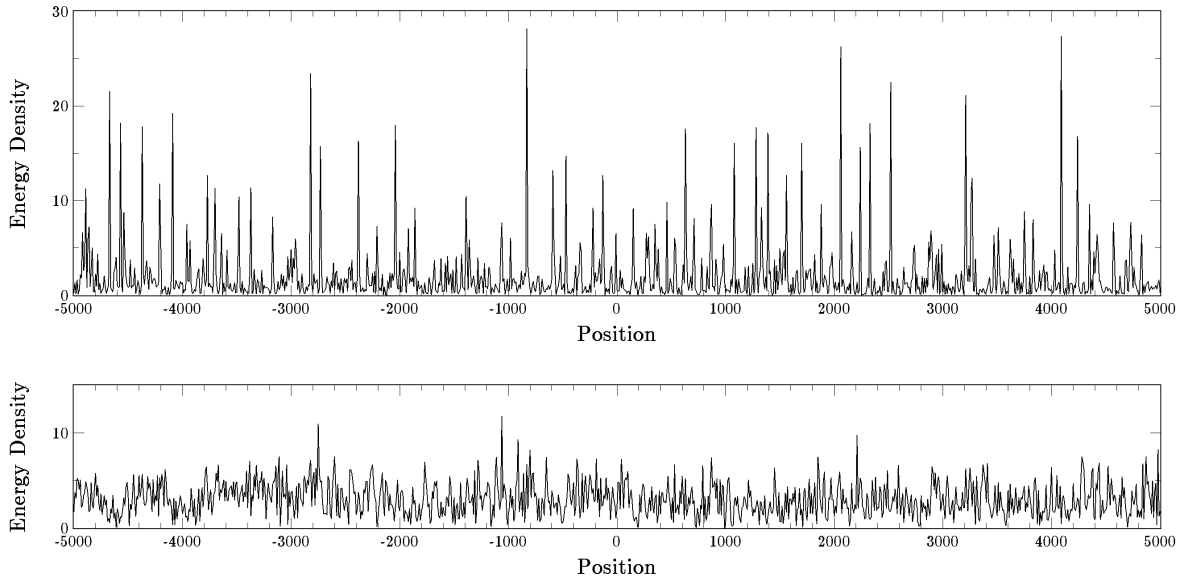


FIGURE 1. Energy density vs. position x for a numerical simulation of (1.1) with periodic boundary conditions and $V(y) = \frac{1}{2}y^2 - \frac{1}{4}y^4 + \frac{1}{6}y^6$, suggesting localized oscillations (top), and a simulation of the same equation, but with a different potential $V(y) = \frac{1}{2}y^2 + \frac{1}{4}y^4 + \frac{1}{6}y^6$, for which oscillons do not occur, shown for comparison (bottom). Oscillons occur for the first potential and not the second for the following physical reason: the $-y^4$ term causes large amplitude perturbations to see a flatter potential and thus oscillate at lower frequency. The low-frequency oscillations (oscillons) decouple from the higher-frequency, lower-amplitude travelling waves that would otherwise cause them to dissipate. (The y^6 term is included to maintain a positive potential and avoid unbounded oscillations). (from a 9/26/05 presentation to DOE by A. Guth, with E. Farhi, R. Rosales, N. Graham, A. Scardicchio, and R. Stowell)

in the chosen units. The system (1.1) is formally Hamiltonian, with time-dependent Hamiltonian density

$$(1.2) \quad \mathcal{H}(t) = \frac{1}{2}e^{-Ht}[(\partial_x u)^2 + \pi^2] + e^{Ht}V(u),$$

where $\pi \equiv e^{Ht}u$ is the canonical momentum that is conjugate to u . That is, we can write the canonical equations:

$$\dot{u} = \frac{\partial \mathcal{H}}{\partial \pi}, \dot{\pi} = -\frac{\partial \mathcal{H}}{\partial u}.$$

Liouville's theorem, which applies even when the Hamiltonian is explicitly time-dependent, precludes the collapse of the whole phase space to a lower-dimensional manifold, so that a standard attractor cannot exist for the oscillon equation. This paper demonstrates that,

by suitably adapting the notions of dissipativity and of *basin of attraction*, an alternative construct, the *pullback attractor*, is able to capture the intuitive notion that oscillons define a restricted portion of the full phase space.

Well-posedness of (1.1), supplemented with ordinary (e.g. periodic or Dirichlet) boundary conditions, is classical. In fact, to this regard (1.1) can be seen as a nonlinear damped wave equation in space dimension 1 with time-dependent speed of propagation e^{-Ht} . It is easy to see that it generates a nonautonomous dynamical system (*process*) on the energy phase space $(u, \partial_t u) \in H^1(0, 1) \times L^2(0, 1)$ (see references). Therefore, the main concern of the paper is the study of the dissipative properties and asymptotic behavior of the process. In particular, we are interested in the construction of some sort of attractor, i.e. a “thin” (compact, possibly finite-dimensional) subset of the phase space which is invariant under the process and embodies the long-term dynamics of the system.

While the theory of attractors for autonomous systems is well-established (see [2, 19, 20, 21, 30] for theoretical background and classical applications), there is less common ground in the nonautonomous case. The *uniform attractor* approach, initiated by Haraux [20], and further developed by Chepyzhov and Vishik [9, 10], relies essentially on the compactness of the nonautonomous terms in the equation, and therefore is not applicable in our case.

On the contrary, the framework of pullback attractors, as developed in [12, 13, 27] (see also [5, 6, 8] for more recent investigations), does not pose essential restrictions (for example, translation boundedness or translation compactness) on the nonautonomous terms and allows for time-dependent limit objects (absorbing families and attractors). Roughly speaking, a pullback attractor $\mathcal{A} = \{\mathcal{A}(t)\}$ is a (time-dependent) family of compact sets which is invariant under the solution operator and attracts all solutions originating sufficiently far in the past. The set $\mathcal{A}(t)$ describes the *regime* of the system at time t ; the solutions starting sufficiently early have forgotten the initial data and, thanks to the invariance property, their future evolution is well described by \mathcal{A} . In contrast to the uniform attractor framework, the sets $\mathcal{A}(t)$ may very well not be uniformly bounded as $t \rightarrow +\infty$.

For our system, the lack of dissipation for the natural energy

$$\mathcal{E}(t) = e^{-2Ht} \int_0^1 |\partial_x u(x, t)|^2 dx + \int_0^1 |\partial_t u(x, t)|^2 dx,$$

due to the singularities of the speed of propagation for $t \rightarrow \pm\infty$ and to the structure of the equation, prevents the existence of a pullback absorbing set in the usual sense. For example, in the linear homogeneous case ($V = 0$), one has the conservation law

$$e^{2Ht} \mathcal{E}(t) = e^{2Hs} \mathcal{E}(s), \quad t, s \in \mathbb{R},$$

so that $\|\partial_x u(t)\|_{L^2}$ approaches a constant depending on the size of the data at time s , as $t \rightarrow +\infty$.

To circumvent these issues, we adopt a new point of view on pullback dissipativity. *The main idea is to restrict the basin of attraction to those families of sets of the phase space whose (time-dependent) energy $\mathcal{E}(t)$, dictated by the problem, is bounded as time goes to $-\infty$.* To this purpose, we describe the solution operator as a family of maps acting on a time-dependent family of spaces X_t . In our problem, the spaces are all the same linear

space with the norms $\|\cdot\|_{X_t}$, $\|\cdot\|_{X_s}$ equivalent for any fixed t, s . However, as it will be clear below, this equivalence blows up as we let $s, t \rightarrow \pm\infty$.

Plan of the paper. In the subsequent Section 2, we describe an abstract framework for dynamical processes on time-dependent spaces and provide the main existence result for time-dependent global attractors.

In Section 3, we formulate the evolution problem generated by eq. (1.1) and provide the main dissipative estimate of the paper. Section 4 is devoted to the construction of the time-dependent attractor for the process corresponding to eq. (1.1). In Section 5 we briefly describe the structure of the attractor and some forward convergence results in the case of the potentials coming from the physics literature. In particular, possible nontriviality of the pullback attractor for very flat potentials, discussed in Subsection 5.1, points to a way in which the oscillon behavior of Figure 1 might be explained in future extensions of the present work. Finally, Section 6 is dedicated to establishing finite dimensional reduction on the time-dependent attractor.

2. ATTRACTORS IN TIME-DEPENDENT SPACES.

As roughly described in the introduction, we need to modify the classical framework of pullback attractors in order to handle evolution problems, like (1.1), where the nonautonomous terms appear at a *functional* level, even acting on the space derivatives of the highest order, and not merely as an external time-dependent forcing. Therefore, the two parameter solution operator will be described by a family of maps acting on a one parameter family of spaces X_t , which we continue to call process.

Process. For $t \in \mathbb{R}$, let X_t be a family of Banach spaces. A (continuous) *process* is a two-parameter family of mappings $\{S(t, s) : X_s \rightarrow X_t\}_{s \leq t}$ with properties

- (i) $S(t, t) = \text{Id}_{X_t}$;
- (ii) $S(t, s) \in \mathcal{C}(X_s, X_t)$;
- (iii) $S(\tau, t)S(t, s) = S(\tau, s)$ for $s \leq t \leq \tau$.

In the concrete case examined in §3-§5, the spaces X_t are all the same linear space with the norms $\|\cdot\|_{X_t}$, $\|\cdot\|_{X_s}$ equivalent for any fixed t, s , whereas the equivalence blows up as we let $s, t \rightarrow \pm\infty$. However, this is not needed for most of the theory we develop hereafter, and the present framework can handle evolution problems in which the spaces $\{X_t\}$ are completely unrelated.

Pullback-bounded family. A family of subsets $\mathcal{B} = \{\mathcal{B}(t) \subset X_t\}_{t \in \mathbb{R}}$ is *pullback-bounded* if¹

$$R(t) = \sup_{s \in (-\infty, t]} \|\mathcal{B}(s)\|_{X_s} < \infty \quad \forall t \in \mathbb{R},$$

i.e. the sets $\mathcal{B}(t) \subset X_t$ are bounded for all times and $\|\mathcal{B}(s)\|_{X_s}$ is bounded as $s \rightarrow -\infty$.

¹Here, for D subset of a Banach space X , $\|D\|_X = \sup_{z \in D} \|z\|_X$

Pullback absorber. A pullback-bounded family $\mathbb{A} = \{\mathbb{A}(t)\}$ is called *pullback absorber* if for every pullback-bounded family \mathcal{B} and for every $t \in \mathbb{R}$ there exists $t_0 = t_0(t) \leq t$ such that

$$S(t, s)\mathcal{B}(s) \subset \mathbb{A}(t), \quad \forall s \leq t_0.$$

Although the restriction of the basin of attraction to parametrized families of sets has been employed before in the standard framework of pullback attractors (see for example [4, 6, 7]), our definitions have a stronger physical connotation. It seems physically reasonable to assume that the collection of “bounded sets”, which usually constitutes the basin of absorption, or attraction of a global attractor, contains only those families with bounded energy, as dictated by the problem.

Time-dependent global attractor. A family of compact subsets $\mathcal{A} = \{\mathcal{A}(t) \subset X_t\}_{t \in \mathbb{R}}$ is called *time-dependent global attractor* if it fulfills the following properties:

- (i) (invariance) $S(t, s)\mathcal{A}(s) = \mathcal{A}(t)$, for every $s \leq t$;
- (ii) (pullback attraction) for every pullback-bounded family \mathcal{B} and every $t \in \mathbb{R}$,²

$$\lim_{s \rightarrow -\infty} \text{dist}_{X_t}(S(t, s)\mathcal{B}(s), \mathcal{A}(t)) = 0.$$

If property (ii) holds uniformly wrt $t \in \mathbb{R}$, \mathcal{A} is a *uniform* time-dependent global attractor.

Remark 2.1. In general, conditions (i)-(ii) are not sufficient for uniqueness of the time-dependent attractor (see [23] for a discussion on this issue and examples). However, if we require in addition

- (iii) \mathcal{A} is a pullback-bounded family,

then there exists at most one family satisfying (i)-(iii), i.e. a pullback-bounded time-dependent global attractor is unique in the class of pullback-bounded families. Indeed, assume that \mathcal{A} and \mathcal{A}' are two pullback-bounded time-dependent global attractors for the process $S(\cdot, \cdot)$. Fix $t \in \mathbb{R}$ and observe that (ii) implies

$$\text{dist}_{X_t}(S(t, s)\mathcal{A}'(s), \mathcal{A}(t)) \rightarrow 0, \quad s \rightarrow -\infty.$$

But $S(t, s)\mathcal{A}'(s) = \mathcal{A}'(t)$, so that $\mathcal{A}'(t) \subset \mathcal{A}(t)$, since $\mathcal{A}(t)$ is closed. Exchanging the roles of $\mathcal{A}(t)$ and $\mathcal{A}'(t)$ we get the reverse inclusion as well, so that $\mathcal{A}(t) = \mathcal{A}'(t)$, and finally $\mathcal{A} = \mathcal{A}'$.

To obtain unconditional (i.e. without assuming (iii)) uniqueness of the time-dependent global attractor, one has to rule out the existence of families of compact, invariant, pullback-attracting sets which are not pullback-bounded: this can be done by establishing that the process has the backward boundedness property.

²For a Banach space X and $A, B \subset X$, the Hausdorff semidistance is defined as

$$\text{dist}_X(A, B) = \sup_{x \in A} \inf_{y \in B} \|y - x\|_X.$$

From the definition, $\text{dist}_X(A, B) = 0$ if and only if A is contained in the closure of B .

Time-dependent ω -limit. Given a family of sets \mathcal{B} , its time-dependent ω -limit is the family $\omega_{\mathcal{B}} = \{\omega_{\mathcal{B}}(t) \subset X_t\}_{t \in \mathbb{R}}$, where $\omega_{\mathcal{B}}(t)$ is defined as

$$\omega_{\mathcal{B}}(t) = \bigcap_{\tau \leq t} \overline{\bigcup_{s \leq \tau} S(t, s)\mathcal{B}(s)},$$

and the above closure is taken in X_t . An equivalent characterization is the following:

$$\omega_{\mathcal{B}}(t) = \{z \in X_t : \exists s_n \rightarrow -\infty, z_n \in \mathcal{B}(s_n) \text{ with } \|S(t, s_n)z_n - z\|_{X_t} \rightarrow 0 \text{ as } n \rightarrow \infty\}.$$

A word of warning. In the remainder of the paper, we will generally omit the phrase *time-dependent* and hence say *global attractor* for *time-dependent global attractor*.

Existence of the global attractor. This subsection is devoted to a result of existence of a (time-dependent) global attractor. We will make use of the Kuratowski measure of noncompactness: if X is a Banach space and $D \subset X$,

$$\alpha(D) = \inf\{d > 0 : D \text{ has a finite cover of balls of } X \text{ of radius less than } d\}.$$

Hereafter we recall some properties of the Kuratowski measure we are going to use, redirecting to [19] for more details and proofs:

(K.1) $\alpha(D) = 0$ if and only if A is compact in X ;

(K.2) $D_1 \subset D_2$ implies $\alpha(D_1) \leq \alpha(D_2)$;

(K.3) $\alpha(D) = \alpha(\overline{D})$;

(K.4) if $r_0 \in \mathbb{R}$ and $\{\mathcal{U}_r\}_{r \geq r_0}$ is a family of nonempty closed subsets of X such that

$$\mathcal{U}_{r_2} \subset \mathcal{U}_{r_1} \quad \forall r_2 > r_1 \geq r_0, \quad \lim_{r \rightarrow +\infty} \alpha(\mathcal{U}_r) = 0,$$

then $\mathcal{U} = \bigcap_{r \geq r_0} \mathcal{U}_r$ is nonempty and compact;

(K.5) let $\{\mathcal{U}_r\}_{r \geq r_0}$ be as in (K.4) and let any two sequences $r_n \rightarrow +\infty$, $x_n \in \mathcal{U}_{r_n}$ be given; then x_n possesses a subsequence converging to some $x \in \mathcal{U}$.

Remark 2.2. The shorthand α_t stands for the Kuratowski measure in the space X_t . We remark that, for fixed $s, t \in \mathbb{R}$, α_s and α_t are equivalent measures of noncompactness whenever there is a Banach space isomorphism between X_s and X_t .

Theorem 2.1. Assume that the process $S(\cdot, \cdot)$ possesses an absorber \mathbb{A} for which

$$(2.1) \quad \lim_{s \rightarrow -\infty} \alpha_t(S(t, s)\mathbb{A}(s)) = 0, \quad \forall t \in \mathbb{R}.$$

Then, $\omega_{\mathbb{A}}$ is a global attractor for $S(\cdot, \cdot)$.

Before the proof of Theorem 2.1, we make some remarks and derive two useful corollaries.

Remark 2.3. We can compare Theorem 2.1 with the previous literature by examining the case $X_t = X$ (the Banach space is fixed), where \mathcal{A} is the usual pullback (global) attractor. The existence result in e.g. [12] relies on the compactness of the process, which is possible for parabolic systems in a bounded domain. For hyperbolic systems, we mention a criterion based on the *pullback asymptotic compactness* of the solution operator ([6], see also [24] for a similar construction in the uniform case), successively employed for example in [32]. In [29, Theorem 3.10], the authors provide an abstract theorem which

relies on the α -contractivity of the process (a property which implies (2.1)), though in their setting a nested pullback absorber (i.e. $\mathbb{A}(s) \subset \mathbb{A}(t)$ when $s \leq t$) is needed.

Remark 2.4. Taking advantage of the same techniques used in [25] for the construction of a global attractor for closed semigroups, the strong continuity assumption $S(t, s) \in \mathcal{C}(X_s, X_t)$ can be relaxed to the weaker

- (iii)' $S(t, s) : X_s \rightarrow X_t$ is a closed operator: given a convergent sequence $x_n \rightarrow x$ in X_s , if $S(t, s)x_n \rightarrow y$ in X_t , then $y = S(t, s)x$.

In particular, (iii)' holds whenever $S(t, s)$ is norm-to-weak continuous.

The following corollary is a (more concrete) reformulation of Theorem 2.1.

Corollary 2.1. *If the process $S(\cdot, \cdot)$ with absorber \mathbb{A} possesses a decomposition*

$$S(t, s)\mathbb{A}(s) = P(t, s) + N(t, s)$$

where

$$\lim_{s \rightarrow -\infty} \|P(t, s)\|_{X_t} = 0, \quad \forall t \in \mathbb{R}$$

and $N(t, s)$ is a compact subset of X_t for all $t \in \mathbb{R}$ and $s \leq t$, then $\mathcal{A}(t) = \omega_{\mathbb{A}}(t)$ is a global attractor for the process $S(\cdot, \cdot)$.

Before proving Theorem 2.1, we dwell on the regularity and uniqueness of the global attractor in a further corollary.

Corollary 2.2. *Let Y_t be a further family of Banach spaces satisfying, for every $t \in \mathbb{R}$,*³

- $Y_t \Subset X_t$;
- denoting with $\mathcal{I}_t : Y_t \rightarrow X_t$ the canonical injection, the maps \mathcal{I}_s are equibounded for $s \leq t$, i.e. $\sup_{s \leq t} \|\mathcal{I}_s\|_{\mathcal{L}(Y_s, X_s)} = C(t) < \infty$;
- closed balls of Y_t are closed⁴ in X_t .

Under the same assumptions as in Corollary 2.1, if in addition

$$\sup_{s \in (-\infty, t]} \|N(t, s)\|_{Y_t} = h(t) < \infty \quad \forall t \in \mathbb{R},$$

then the global attractor \mathcal{A} is a pullback-bounded family, and henceforth unique in the sense of Remark 2.1. Furthermore, it satisfies

$$\|\mathcal{A}(t)\|_{Y_t} \leq h(t) \quad \forall t \in \mathbb{R}.$$

Proof. Fix $t \in \mathbb{R}$ and $z \in \mathcal{A}(t)$. By definition, there exists sequences $s_n \rightarrow -\infty$, $z_n \in \mathbb{A}(s_n)$ such that $\|S(t, s_n)z_n - z\|_{X_t} \rightarrow 0$ as $n \rightarrow \infty$. Using the decomposition of Corollary 2.1,

$$S(t, s_n)z_n = P_{z_n}(t, s_n) + N_{z_n}(t, s_n),$$

with $P_{z_n}(t, s_n) \in P(t, s_n)$, and $N_{z_n}(t, s_n) \in N(t, s_n)$. In particular $\|N_{z_n}(t, s_n)\|_{Y_t} \leq h(t)$, i.e. the sequence $N_{z_n}(t, s_n)$ is contained in the closed ball of Y_t with radius $h(t)$, which we call B_t . Now, using $\|P(t, s)\|_{X_t} \rightarrow 0$ as $s \rightarrow -\infty$

$$\|N_{z_n}(t, s_n) - z\|_{X_t} \leq \|S(t, s_n)z_n - z\|_{X_t} + \|P_{z_n}(t, s_n)\|_{X_t} \rightarrow 0, \quad n \rightarrow \infty.$$

³With $Y \Subset X$ we indicate compact injection of the Banach space Y into the Banach space X .

⁴For example, this holds when Y_t is reflexive and compactly embedded into X_t .

Therefore, z is an accumulation point of B_t (in the topology of X_t). By assumption, B_t is closed in X_t , so that $z \in B_t$ as well.

This establishes that $\mathcal{A}(t) \subset B_t$, i.e. $\|\mathcal{A}(t)\|_{Y_t} \leq h(t)$ for every $t \in \mathbb{R}$, which in turn yields that \mathcal{A} is a pullback-bounded family in Y_t . The second assumption yields the existence of $C = C(t) > 0$ such that

$$\|\mathcal{A}(s)\|_{X_s} \leq C(t)h(s), \quad \forall s \leq t.$$

Taking supremum over $s \leq t$, since h is increasing by definition, we obtain

$$\sup_{s \in (-\infty, t]} \|\mathcal{A}(s)\|_{X_s} \leq C(t)h(t),$$

i.e. \mathcal{A} is a pullback-bounded family. Uniqueness then follows from Remark 2.1. \square

Proof of Theorem 2.1. We will prove that $\omega_{\mathbb{A}}$, as defined above, is a family of compact sets satisfying (i)-(ii), and hence, is a global attractor for the process $S(t, s)$. We split the proof into two parts.

\diamond Compactness and attraction property of $\omega_{\mathbb{A}}(t)$.

Let $t \in \mathbb{R}$ be fixed, and $\varepsilon > 0$ be arbitrary. By assumption, there exists $t_0 \leq t$ such that $\alpha_t(S(t, s)\mathbb{A}(s)) < \varepsilon$ whenever $s \leq t_0$. Then, since \mathbb{A} is an absorber, we can find $s_0 \leq t_0$ satisfying

$$S(t_0, s)\mathbb{A}(s) \subset \mathbb{A}(t_0) \quad \forall s \leq s_0.$$

Therefore, for $\tau \leq s_0$,

$$\mathcal{U}_\tau = \bigcup_{s \leq \tau} S(t, s)\mathbb{A}(s) = \bigcup_{s \leq \tau} S(t, t_0)S(t_0, s)\mathbb{A}(s) \subset \bigcup_{s \leq \tau} S(t, t_0)\mathbb{A}(t_0) = S(t, t_0)\mathbb{A}(t_0),$$

which yields $\alpha_t(\mathcal{U}_\tau) < \varepsilon$ whenever $\tau \leq s_0$. Therefore the sets $\overline{\mathcal{U}_\tau}$ are closed subsets of X_t , nested as $\tau \rightarrow -\infty$ and

$$\lim_{\tau \rightarrow -\infty} \alpha_t(\overline{\mathcal{U}_\tau}) = 0.$$

It follows by property (K.4) of the Kuratowski measure that $\omega_{\mathbb{A}}(t) = \bigcap_{\tau \leq t} \overline{\mathcal{U}_\tau}$ is nonempty and compact.

We now prove that $\omega_{\mathbb{A}}(t)$ is attracting in the sense of Definition 1.4-(ii). Suppose not, then there exists $t \in \mathbb{R}$, a pullback-bounded family \mathcal{B} , $s_n \rightarrow -\infty$, $z_n \in \mathcal{B}(s_n)$ and $\delta > 0$ such that

$$(2.2) \quad \inf_{z \in \omega_{\mathbb{A}}(t)} \|S(t, s_n)z_n - z\|_{X_t} > \delta.$$

Extract a subsequence $\{s_{n_k}\}$ from $\{s_n\}$ as follows: given s_{n_1}, \dots, s_{n_k} , choose $s_{n_{k+1}} \leq s_{n_k}$ such that $S(s_{n_k}, s_{n_{k+1}})\mathcal{B}(s_{n_{k+1}}) \subset \mathbb{A}(s_{n_k})$. Now observe that

$$S(t, s_{n_{k+1}})z_{n_{k+1}} = S(t, s_{n_k})S(s_{n_k}, s_{n_{k+1}})z_{n_{k+1}} = S(t, s_{n_k})w_k,$$

with $w_k = S(s_{n_k}, s_{n_{k+1}})z_{n_{k+1}} \in \mathbb{A}(s_{n_k})$, i.e.

$$S(t, s_{n_{k+1}})z_{n_{k+1}} \in \mathcal{U}_{s_{n_k}}.$$

By property (K.5) of the Kuratowski measure, the sequence $S(t, s_{n_{k+1}})z_{n_{k+1}}$ has an accumulation point in $\omega_{\mathbb{A}}(t)$, which contradicts (2.2). Therefore, the first step is complete.

◇ Invariance (in the sense of Definition 1.4(i)) of $\omega_{\mathbb{A}}(t)$.

Let $t \geq s$. We aim to prove that $\omega_{\mathbb{A}}(t) = S(t, s)\omega_{\mathbb{A}}(s)$.

We first deal with the inclusion $\omega_{\mathbb{A}}(t) \supset S(t, s)\omega_{\mathbb{A}}(s)$, which is easier. Let $z \in \omega_{\mathbb{A}}(s)$, then, there exist sequences $s_n \rightarrow -\infty$, $z_n \in \mathbb{A}(s_n)$, such that

$$(2.3) \quad \|S(s, s_n)z_n - z\|_{X_s} \rightarrow 0, \quad n \rightarrow \infty.$$

Now, extract again a subsequence $\{s_{n_k}\}$ as follows: given s_{n_1}, \dots, s_{n_k} , choose $s_{n_{k+1}} \leq s_{n_k}$ such that $S(s_{n_k}, s_{n_{k+1}})\mathbb{A}(s_{n_{k+1}}) \subset \mathbb{A}(s_{n_k})$. Hence, we set

$$w = S(t, s)z \in S(t, s)\omega_{\mathbb{A}}(s), \quad w_k = S(s_{n_k}, s_{n_{k+1}})z_{n_{k+1}} \in \mathbb{A}(s_{n_k}).$$

Thus, we also have

$$\begin{aligned} \|S(t, s_{n_k})w_k - w\|_{X_t} &= \|S(t, s)S(s, s_{n_k})w_k - S(t, s)z\|_{X_t} \\ &= \|S(t, s)S(s, s_{n_{k+1}})z_{n_{k+1}} - S(t, s)z\|_{X_t} \rightarrow 0 \end{aligned}$$

as $k \rightarrow \infty$, using (2.3) and the continuity of $S(t, s)$. This establishes that $w = S(t, s)z \in \omega_{\mathbb{A}}(t)$ and henceforth $\omega_{\mathbb{A}}(t) \supset S(t, s)\omega_{\mathbb{A}}(s)$ as claimed.

We now turn to the reverse inclusion $\omega_{\mathbb{A}}(t) \subset S(t, s)\omega_{\mathbb{A}}(s)$. Let $z \in \omega_{\mathbb{A}}(t)$ be arbitrary, and choose sequences $s_n \rightarrow -\infty$, $z_n \in \mathbb{A}(s_n)$ such that $s_n \leq s$ for all n and

$$(2.4) \quad \|S(t, s_n)z_n - z\|_{X_t} \rightarrow 0, \quad n \rightarrow \infty.$$

Using the attraction property (ii), proven in the first part,

$$\lim_{n \rightarrow \infty} \inf_{w \in \omega_s(\mathbb{A})} \|S(s, s_n)z_n - w\|_{X_s} = 0,$$

so that there exists a sequence $w_n \in \omega_{\mathbb{A}}(s)$ satisfying

$$\lim_{n \rightarrow \infty} \|S(s, s_n)z_n - w_n\|_{X_s} = 0$$

By compactness of $\omega_{\mathbb{A}}(s)$, we have $w_n \rightarrow w \in \omega_{\mathbb{A}}(s)$ in X_s up to a subsequence. This yields $S(s, s_n)z_n \rightarrow w$, in X_s , and, by continuity of $S(t, s)$, $S(t, s_n)z_n \rightarrow S(t, s)w$ (in X_t) as well, i.e. $z = S(t, s)w$, which completes the proof of the second inclusion, and in turn, of Theorem 2.1. \square

3. THE OSCILLON EQUATION WITH A GENERAL POTENTIAL

3.1. Setting of the problem. In the following, $|\cdot|$ and $\langle \cdot, \cdot \rangle$ denote respectively the standard norm and scalar product on $L^2(0, 1)$; A denotes $-\Delta$ on $(0, 1)$ with periodic boundary conditions.

The symbols c and \mathcal{Q} will stand respectively for a generic positive constant and a generic positive increasing continuous function, which may be different in different occurrences; when an index is added, (e.g. c_0, \mathcal{Q}_0), the positive constant (resp. function) is meant to be specific. Similarly, the symbols Λ, Λ_i will denote certain energy-like functionals occurring in the proofs. Finally, in most of the proofs we adopt the shorthand $\varpi(t) := e^{-2Ht}$.

We study the *oscillon equation* in space dimension $n = 1$ with periodic boundary conditions

$$(P) \quad \begin{cases} \partial_{tt}u(t) + H\partial_t u(t) + e^{-2Ht}Au(t) + \varphi(u(t)) = 0, & t \geq s, \\ u(s) = u_0 \in H_{\text{per}}^1(0, 1), \partial_t u(s) = v_0 \in L^2(0, 1). \end{cases}$$

Here, (except at the end of Section 4) we consider a general nonlinear potential $V \in \mathcal{C}^2(\mathbb{R})$, such that $V(0) = 0$, and $\varphi = V'$ satisfies the following assumptions:

(H0) $\varphi(0) = 0$;

(H1) there exist $a_0, a_2 > 0$, $a_1, a_3 \geq 0$, $q \geq 2$ such that

$$a_0|y|^{q-2} - a_1 \leq \varphi'(y) \leq a_2|y|^{q-2} + a_3.$$

In addition, when $q = 2$ (sublinear case), we assume $a_0 > a_1$.

Note that assumptions (H0)-(H1), as well as assumption (6.4) in Theorem 6.1, are satisfied by any (non-constant) polynomial V with $V(0) = 0$ and positive leading coefficient. In the class of polynomials, only the above mentioned polynomials satisfy (H0)-(H1).

Also note that, since $V(0) = \varphi(0) = 0$, two consecutive integrations of (H1) yield

$$(3.1) \quad \frac{a_0}{q(q-1)}|y|^q - \frac{a_1}{2}y^2 \leq V(y) \leq \frac{a_2}{q(q-1)}|y|^q + \frac{a_3}{2}y^2.$$

Moreover, integrating by parts and using (H0)-(H1), we have, say for $y \geq 0$,

$$\begin{aligned} y\varphi(y) - V(y) &= \int_0^y \sigma \varphi'(\sigma) d\sigma \\ &\geq \int_0^y \sigma (a_0|\sigma|^{q-2} - a_1) d\sigma \\ &\geq \frac{a_0}{q}|y|^q - \frac{a_1}{2}y^2, \end{aligned}$$

and similarly for $y < 0$, so that

$$(3.2) \quad y\varphi(y) \geq V(y) + \frac{a_0}{q}|y|^q - \frac{a_1}{2}y^2 \geq V(y) - c_0,$$

for some $c_0 \geq 0$ depending only on a_0, a_1, q . In particular, we can take $c_0 = 0$ whenever $a_1 = 0$.

Remark 3.1. For the physically interesting potentials

$$V_+(y) = \frac{y^2}{2} + \frac{y^4}{4} + \frac{y^6}{6}, \quad V_-(y) = \frac{y^2}{2} - \frac{y^4}{4} + \frac{y^6}{6},$$

the nonlinearities read $\varphi_{\pm}(y) = y \pm y^3 + y^5$. Noting that $3y^2 < 1 + 4y^4$, it is immediate to observe that φ_+ fulfills both (H0) and (H1), for $q = 6$ and e.g. $a_0 = 5, a_1 = 0, a_2 = 6, a_3 = 2$. The same observation yields that, for φ_- , (H0) and (H1) are satisfied with $q = 6, a_0 = 1, a_1 = 0, a_2 = 5, a_3 = 1$.

We set

$$\mathcal{V}(u) = \int_0^1 V(u(x)) dx.$$

In view of (3.1), $\mathcal{V}(u)$ is well defined for every $u \in L^q(0, 1)$, and

$$(3.3) \quad b_0(\|u\|_{L^q}^q + |u|^2) - b_1 \leq \mathcal{V}(u) \leq b_2(\|u\|_{L^q}^q + |u|^2),$$

with $b_0, b_2 > 0$ and $b_1 \geq 0$ depending only on the a_i ($i = 0, \dots, 3$) and q ; in particular, $b_1 = 0$ whenever $a_1 = 0$.

Problem (P) is rewritten in our abstract framework as follows. The phase spaces are the Banach spaces

$$X_t = H_{\text{per}}^1(0, 1) \times L^2(0, 1) \quad \text{with norms} \quad \|(u, v)\|_{X_t} = e^{-Ht}|A^{1/2}u| + \|u\|_{L^q} + |v|.$$

For simplicity, we set $X = X_0$. It is clear that the spaces X_t are all the same as linear spaces; given $z \in X$, its injection in X_t will still be denoted as z . Since for every $t \in \mathbb{R}$, $z \in X$

$$\min\{e^{-Ht}, 1\}\|z\|_X \leq \|z\|_{X_t} \leq \max\{e^{-Ht}, 1\}\|z\|_X$$

the norms $\|\cdot\|_{X_s}, \|\cdot\|_{X_t}$ are equivalent for any fixed $t, s \in \mathbb{R}$. Again, we stress how the equivalence constants blow up when $t \rightarrow \pm\infty$.

Remark 3.2. For some of the proofs below, it will be convenient to use the natural energy of the problem at time t

$$(3.4) \quad \mathcal{E}_{X_t}(u, v) = e^{-2Ht}|A^{1/2}u|^2 + \frac{2}{q}\|u\|_{L^q}^q + |u|^2 + |v|^2$$

in place of the X_t -norm. It is easy to see that the energy $\mathcal{E}_{X_t}(\cdot)$ is equivalent to the norm $\|\cdot\|_{X_t}$, in the following sense:

· a family $\mathcal{B} = \{\mathcal{B}(t) \subset X_t\}$ is pullback-bounded if and only if

$$\sup_{s \in (-\infty, t]} \sup_{z \in \mathcal{B}(s)} \mathcal{E}_{X_s}(z) < \infty \quad \forall t \in \mathbb{R};$$

· a sequence $\{z_n\} \subset X_t$ converges to $z \in X_t$ if and only if $\mathcal{E}_{X_t}(z_n - z)$ converges to zero.

3.2. Well-posedness and dissipativity. Under the assumptions made in the preceding section, we are able to obtain a well-posedness result.

Theorem 3.1. *Problem (P) generates a strongly continuous process $S(t, s) : X_s \rightarrow X_t$, satisfying the following dissipative estimate: for $z \in X$,*

$$(3.5) \quad \mathcal{E}_{X_t}(S(t, s)z) \leq K_0(\mathcal{E}_{X_s}(z))e^{-\mu(t-s)} + K_1, \quad \forall t \geq s,$$

with $K_1 = 4c_1^{-1}(c_0 + b_1)$. The positive constants c_1, K_0, μ are explicitly defined in the proof, and depend only on the physical parameters H, a_i and q .

Furthermore, the following additional continuous dependence property holds: for every pair of initial conditions $z^i \in X$ ($i = 1, 2$) with $\mathcal{E}_{X_s}(z^i) \leq R$ and every $t \geq s$, we have

$$(3.6) \quad \mathcal{E}_{X_t}[S(t, s)z^1 - S(t, s)z^2] \leq \mathcal{Q}_1(R) \exp((t-s) + e^{H\sigma t} - e^{H\sigma s}) \mathcal{E}_{X_s}[z^1 - z^2],$$

where $\sigma = \frac{q}{2} - 1$ and \mathcal{Q}_1 is specified in the proof.

Proof. We begin with the formal derivation of estimate (3.5). Hereafter, $(u(t), \partial_t u(t))$ denotes the solution to (P) with initial time $s \in \mathbb{R}$ and initial condition $z = (u_0, v_0) \in X$, which we assume to be sufficiently regular.

A multiplication of (P) by $\partial_t u$ entails

$$(3.7) \quad \frac{d}{dt} [\varpi |A^{1/2} u|^2 + |\partial_t u|^2 + 2\mathcal{V}(u)] + 2H[\varpi |A^{1/2} u|^2 + |\partial_t u|^2] = 0,$$

while multiplying (P) by u and then using (3.2) yields

$$(3.8) \quad \frac{d}{dt} [H|u|^2 + 2\langle \partial_t u, u \rangle] + 2\varpi |A^{1/2} u|^2 - 2|\partial_t u|^2 = -2\langle \varphi(u), u \rangle \leq -2\mathcal{V}(u) + 2c_0.$$

For $0 < \nu \leq \frac{H}{4}$ to be determined later, we add (3.7) to ν -times (3.8). Setting

$$\begin{aligned} \Lambda_1 &= \varpi |A^{1/2} u|^2 + |\partial_t u|^2 + 2\mathcal{V}(u) + \nu(H|u|^2 + 2\langle \partial_t u, u \rangle), \\ \Lambda_\star &= (H + \nu)\varpi |A^{1/2} u|^2 + (H - 3\nu)|\partial_t u|^2 - \nu^2 H|u|^2 - 2\nu^2 \langle \partial_t u, u \rangle, \end{aligned}$$

we obtain

$$(3.9) \quad \frac{d}{dt} \Lambda_1 + \nu \Lambda_1 + \Lambda_\star + H[\varpi |A^{1/2} u|^2 + |\partial_t u|^2] \leq 2\nu c_0.$$

Then, we take advantage of (3.3) to get the bound

$$(3.10) \quad c_1 \mathcal{E}_{X_t}[(u(t), \partial_t u(t))] - 2b_1 \leq \Lambda_1(t) \leq c_2 \mathcal{E}_{X_t}[(u(t), \partial_t u(t))]$$

with $c_1 = \min\{qb_0, 1\}/2$ and c_2 a positive constant depending (increasingly) on H and b_2 . Hence, exploiting the left-hand inequality of (3.10) to control $|u|^2$ from above, we find

$$\begin{aligned} \Lambda_\star &\geq (H - 3\nu - \nu^2)|\partial_t u|^2 - \nu^2(H + 1)|u|^2 \\ &\geq -\nu^2(H + 1)|u|^2 \geq -\nu^2(H + 1)c_1^{-1}\Lambda_1 - 2\nu^2 b_1 c_1^{-1} \\ &\geq -\frac{\nu}{2}\Lambda_1 - \nu b_1, \end{aligned}$$

provided we restrict ourselves to $\nu \leq 2\mu = \min\{1, H/4, (4H + 4)^{-1}c_1\}$. Therefore, writing (3.9) for $\nu = 2\mu$ and dropping the rightmost summand on the lhs yields

$$(3.11) \quad \frac{d}{dt} \Lambda_1 + \mu \Lambda_1 \leq 2\mu(2c_0 + b_1).$$

Multiplying the above inequality by $e^{\mu t}$ and integrating between s and t , we obtain

$$(3.12) \quad \Lambda_1(t) \leq \Lambda_1(s)e^{-\mu(t-s)} + 2(2c_0 + b_1);$$

an exploitation of (3.10) then leads to

$$\begin{aligned} (3.13) \quad c_1 \mathcal{E}_{X_t}[(u(t), \partial_t u(t))] &\leq \Lambda_1(t) + 2b_1 \\ &\leq \Lambda_1(s)e^{-\mu(t-s)} + 4(c_0 + b_1) \\ &\leq c_2 \mathcal{E}_{X_s}(z)e^{-\mu(t-s)} + 4(c_0 + b_1) \end{aligned}$$

which is (3.5), with $K_0 = c_2/c_1$, $K_1 = 4c_1^{-1}(c_0 + b_1)$. Note that, like b_1 and c_0 , $K_1 = 0$ when $a_1 = 0$ in (H1).

Having at our disposal (3.5), global existence of (weak) solutions $(u(t), \partial_t u(t))$ to Problem (P) is obtained by means of a standard Galerkin scheme. The solutions we obtain in this way satisfy, on any interval (s, t) , $-\infty < s < t < +\infty$,

$$u \in L^\infty(s, t; H_{\text{per}}^1(0, 1)) \cap L^q(s, t; L^q(0, 1)), \quad \partial_t u \in L^\infty(s, t; L_{\text{per}}^2(0, 1)).$$

Replacing L^∞ on (s, t) with continuity on $[s, t]$ requires some additional work, as explained in [30, Section II.4].

Uniqueness of solutions, and therefore generation of the process $S(t, s)$ will then follow once the continuous dependence estimate (3.6) is established.

To prove (3.6), for $i = 1, 2$, let $z^i = (u_0^i, v_0^i) \in X$ with $\|z^i\|_{X_s} \leq R$. Accordingly, call $(u^i(t), \partial_t u^i(t))$ the solution corresponding to initial datum z^i , prescribed at time $s \in \mathbb{R}$. Preliminarily, we recall that the dissipative estimate (3.5) can be rewritten as

$$(3.14) \quad \mathcal{E}_{X_t}[(u^i(t), \partial_t u^i(t))] \leq K_0 R + K_1 := \mathcal{Q}(R), \quad \forall t \geq s.$$

Then, we observe that the difference

$$\bar{z}(t) = (u^1(t), \partial_t u^1(t)) - (u^2(t), \partial_t u^2(t)) = (\bar{u}(t), \partial_t \bar{u}(t))$$

fulfills the Cauchy problem on $(s, +\infty)$

$$\begin{cases} \partial_{tt} \bar{u} + H \partial_t \bar{u} + \varpi A \bar{u} + \bar{u} |\bar{u}|^{q-2} + \bar{u} = \bar{u} + \varphi(u^2) - \varphi(u^1) + \bar{u} |\bar{u}|^{q-2}, \\ \bar{z}(s) = z^1 - z^2. \end{cases}$$

Assuming $(\bar{u}, \partial_t \bar{u})$ sufficiently smooth, we multiply the above equation by $\partial_t \bar{u}$ and obtain the differential inequality

$$(3.15) \quad \frac{d}{dt} \mathcal{E}_{X_t}(\bar{z}) \leq 2 \langle \bar{u} + \varphi(u^2) - \varphi(u^1) + \bar{u} |\bar{u}|^{q-2}, \partial_t \bar{u} \rangle.$$

The first term in the rhs is easily estimated by $2|\bar{u}||\partial_t \bar{u}|$. Regarding the second, we exploit the Agmon inequality to obtain the bound

$$\|u^i\|_{L^\infty} \leq c(|u^i| + |u^i|^{\frac{1}{2}} |A^{1/2} u^i|^{\frac{1}{2}}), \quad i = 1, 2,$$

so that, using (H1), we estimate

$$\begin{aligned} 2 \langle \varphi(u^2) - \varphi(u^1), \partial_t \bar{u} \rangle &= -2 \langle \varphi'(\xi) \bar{u}, \partial_t \bar{u} \rangle \\ &\leq c(1 + \|u^1\|_{L^\infty}^{q-2} + \|u^2\|_{L^\infty}^{q-2}) |\bar{u}| |\partial_t \bar{u}| \\ &\leq c \left(1 + \sum_{i=1,2} (|u^i| + |u^i|^{\frac{1}{2}} |A^{1/2} u^i|^{\frac{1}{2}})^{q-2} \right) |\bar{u}| |\partial_t \bar{u}|. \end{aligned}$$

Treating $|\bar{u}|^{q-2}$ as done above for φ' yields the similar control

$$2 \langle \bar{u} |\bar{u}|^{q-2}, \partial_t \bar{u} \rangle \leq c \left(1 + \sum_{i=1,2} (|u^i| + |u^i|^{\frac{1}{2}} |A^{1/2} u^i|^{\frac{1}{2}})^{q-2} \right) |\bar{u}| |\partial_t \bar{u}|.$$

Recalling (3.14) and the obvious bounds

$$|u^i(t)| \leq \mathcal{E}_{X_t}(u^i(t), \partial_t u^i(t))^{1/2}, \quad |A^{1/2} u^i(t)|^{\frac{1}{2}} \leq e^{\frac{Ht}{2}} \mathcal{E}_{X_t}(u^i(t), \partial_t u^i(t))^{1/4},$$

we have the estimate

$$\sum_{i=1,2} (|u^i(t)| + |u^i(t)|^{\frac{1}{2}} |A^{1/2} u^i(t)|^{\frac{1}{2}})^{q-2} \leq \mathcal{Q}(R)^\sigma (1 + e^{\sigma H t}),$$

where we have set $\sigma = \frac{q}{2} - 1$, so that the controls on the rhs of (3.15) may be summarized as

$$(3.16) \quad \frac{d}{dt} \mathcal{E}_{X_t}(\bar{z}(t)) \leq c \mathcal{Q}(R)^\sigma (1 + e^{\sigma H t}) \mathcal{E}_{X_t}(\bar{z}(t)).$$

We then apply Gronwall's lemma on (s, t) to get

$$\mathcal{E}_{X_t}(\bar{z}(t)) \leq \mathcal{Q}_1(R) \exp((t - s) + e^{H\sigma t} - e^{H\sigma s}) \mathcal{E}_{X_s}(z^1 - z^2),$$

where $\mathcal{Q}_1(R) = c \exp(\mathcal{Q}(R)^\sigma)$, as claimed in (3.6), so that the proof is complete. \square

Remark 3.3. Note that, in the proof of Theorem 3.1, we have assumed $(u, \partial_t u)$ and $(\bar{u}, \partial_t \bar{u})$ to be smooth enough to derive the estimates (3.7) and (3.15), which led us to (3.5) and (3.6). These energy inequalities can be made rigorous by deriving them for the corresponding Galerkin approximation and then passing to the (lower) limit. Energy equalities (which were not needed here) can be obtained by the regularization procedure described in [30, Lemma II.4.1].

Finally, the derivation of the a priori estimate (3.9), based on adding eq. (1.1) multiplied by u to ν -times eq. (1.1) multiplied by $\partial_t u$ is reminiscent of [17, 30], where we multiply (1.1) by $u + \nu \partial_t u$. However, here we work with the original variables $u, \partial_t u$ in place of $u, u + \nu \partial_t u$ as in [17, 30].

Using the dissipative estimate (3.5), we explicitly construct an absorber for the process generated by (P).

Theorem 3.2. *There exists $R_{\mathbb{A}} = R_{\mathbb{A}}(H, a_i) > 0$ such that the family*

$$(3.17) \quad \mathbb{A} = \{\mathbb{A}(t) = \{z \in X_t : \mathcal{E}_{X_t}(z) \leq R_{\mathbb{A}}\}\}$$

is an absorber for the process $S(\cdot, \cdot)$.

Proof. Let \mathcal{B} be a pullback-bounded family and, for $t \in \mathbb{R}$, let

$$R(t) = \sup_{s \in (-\infty, t]} \mathcal{E}_{X_s}[\mathcal{B}(s)].$$

Observe that Remark 3.2 guarantees $R(t) < \infty$ for every $t \in \mathbb{R}$. Estimate (3.5) then reads

$$\mathcal{E}_{X_t}(S(t, s)z) \leq K_0 \mathcal{E}_{X_s}(z) e^{-\mu(t-s)} + K_1 \leq K_0 R(t) e^{-\mu(t-s)} + K_1 \leq 1 + 2K_1$$

for every $z \in \mathcal{B}(s)$, provided that

$$s \leq t_0 = t - \max\{0, \mu^{-1} \log \frac{K_0 R(t)}{1+K_1}\}.$$

Taking the supremum over $z \in \mathcal{B}(s)$, we obtain

$$\mathcal{E}_{X_t}[S(t, s)\mathcal{B}(s)] \leq 1 + 2K_1, \quad \forall s \leq t_0,$$

which, setting $R_{\mathbb{A}} = 1 + 2K_1$, reads exactly $S(t, s)\mathcal{B}(s) \subset \mathbb{A}(t)$ whenever $s \leq t_0$. \square

4. THE GLOBAL PULLBACK ATTRACTOR.

We now devote ourselves to the construction of a global attractor (in the sense specified in Section 2) for the oscillon equation. Existence of an attractor, uniqueness, and regularity property are specified in the main theorem below.

Theorem 4.1. *The family $\mathcal{A}(t) = \omega_{\mathbb{A}}(t)$ is the unique (in the sense of Remark 2.1) global attractor of the process $S(\cdot, \cdot)$ generated by (P). Moreover, introducing the family of Banach spaces ($t \in \mathbb{R}$)*

$$Y_t = H_{\text{per}}^2(0, 1) \times H^1(0, 1), \quad \|(u, v)\|_{Y_t} = e^{-Ht}|Au| + \|u\|_{L^q} + |A^{1/2}v| + |v|$$

we have

$$\mathcal{A}(t) \subset Y_t, \quad \|\mathcal{A}(t)\|_{Y_t}^2 \leq h(t), \quad \forall t \in \mathbb{R},$$

where $h(t)$ is a continuous increasing function of t , depending on $R_{\mathbb{A}}$, and is defined below.

Remark 4.1. Observe that the injections $\mathcal{I}_t : Y_t \rightarrow X_t$ satisfy

$$\|\mathcal{I}_t\|_{\mathcal{L}(Y_t, X_t)} \leq \max\{1, \lambda_1^{-1}\}$$

(here, λ_1 stands for the Poincaré constant on $(0, 1)$). Hence, the estimate of Theorem 4.1 implicitly asserts that \mathcal{A} is a pullback-bounded family in X_t . Then, the invariance of \mathcal{A} implies that $\mathcal{A}(t) \subset \mathbb{A}(t)$ for every $t \in \mathbb{R}$; more explicitly,

$$(4.1) \quad \sup_{(u,v) \in \mathcal{A}(t)} [e^{-2Ht}|A^{1/2}u|^2 + \|u\|_{L^q}^q + |v|^2] \leq R_{\mathbb{A}}, \quad \forall t \in \mathbb{R}.$$

Remark 4.2. Note that the Hamiltonian structure of the oscillon system is irrelevant to the existence of a global attractor. Such structure exists for any potential V or for $V = 0$. But the proof of Theorem 4.1 depends on a potential V with the property (3.2), through Lemma 4.1 below.

We turn to the proof of Theorem 4.1. We will work throughout with

$$z = (u_0, v_0) \in \mathbb{A}(s);$$

until the end of the section, the generic constants $c > 0$ appearing depend only on $R_{\mathbb{A}}$, whose dependence on the physical parameters of the problem has been specified earlier. Hence, the estimate (3.5) now reads

$$(4.2) \quad \mathcal{E}_{X_t}(S(t, s)z) \leq K_0 R_{\mathbb{A}} + K_1 := c, \quad \forall s \in \mathbb{R}, t \geq s.$$

Now, with the aim of using Corollary 2.1, we perform a suitable decomposition of the solution of Problem (P). We set

$$(u(t), \partial_t u(t)) = S(t, s)z = P_z(t, s) + N_z(t, s) = (p(t), \partial_t p(t)) + (n(t), \partial_t n(t)),$$

where

$$(4.3) \quad \begin{cases} \partial_{tt}p(t) + H\partial_t p(t) + e^{-2Ht}Ap(t) + \varphi_*(p(t)) = 0, & t \geq s, \\ p(s) = u_0, \partial_t p(s) = v_0, \end{cases}$$

$$(4.4) \quad \begin{cases} \partial_{tt}n(t) + H\partial_t n(t) + e^{-2Ht}An(t) = \varphi_*(p(t)) - \varphi(u(t)), & t \geq s, \\ n(s) = 0, \partial_t n(s) = 0. \end{cases}$$

and $\varphi_*(y) = y + |y|^{q-2}y$.

Lemma 4.1. *There exists $K_2, \mu_1 > 0$ so that*

$$\mathcal{E}_{X_t}(P_z(t, s)) \leq K_2 R_{\mathbb{A}} e^{-\mu_1(t-s)} \quad \forall t \in \mathbb{R}, s \leq t.$$

Proof. We peruse the proof of (3.5), Theorem 3.1, replacing φ with φ_* . In this case (H1) holds with $a_1 = 0$, and the corresponding potential $V_*(y) = y^2/2 + |y|^q/q$ satisfies (3.2) with $c_0 = 0$, and (3.3) with (e.g.) $b_0 = 1/q, b_2 = 1$, and $b_1 = 0$. Hence $K_1 = 0$ in (3.5), which is exactly the claimed estimate. Observe that the constants μ_1 and K_2 can be explicitly computed. \square

Lemma 4.2. *There exists a continuous positive increasing function h such that*

$$(4.5) \quad \sup_{s \in (-\infty, t]} \sup_{z \in \mathbb{A}(s)} \|N_z(t, s)\|_{Y_t}^2 \leq h(t) \quad \forall t \in \mathbb{R}.$$

Proof. We first observe that $n(t) = u(t) - p(t)$, so that, using (4.2) and Lemma 4.1,

$$(4.6) \quad \|n(t)\|_{L^q}^q + e^{-2Ht}|A^{1/2}n(t)|^2 + |\partial_t n(t)|^2 \leq c.$$

Therefore, we are only left to control the higher-order seminorms appearing in $\|N_z(t, s)\|_{Y_t}$. To this aim, assuming $(n, \partial_t n)$ is sufficiently regular, we multiply the equation (4.4) by $A\partial_t n$, obtaining for the functional

$$\Lambda_2 = \varpi |An|^2 + |A^{1/2}\partial_t n|^2$$

the differential equation

$$\frac{d}{dt}\Lambda_2 + 2H\Lambda_2 = -2\langle \varphi(u) - \varphi_*(p), A\partial_t n \rangle.$$

Using (H1) with the Agmon inequality, and taking advantage of (4.2) in the last inequality yields

$$\begin{aligned} -2\langle \varphi(u), A\partial_t n \rangle &= -2\langle \varphi'(u)\partial_x u, A^{1/2}\partial_t n \rangle \\ &\leq c(1 + |u|^{q-2}|A^{1/2}u|^{q-2})|A^{1/2}u|^2 + H|A^{1/2}\partial_t n|^2 \\ &\leq c(1 + \|u\|_{L^q}^{q-2})|A^{1/2}u|^q + H|A^{1/2}\partial_t n|^2 + c \\ &\leq c\varpi^{-q/2} + H|A^{1/2}\partial_t n|^2 + c. \end{aligned}$$

Similarly,

$$\begin{aligned} 2\langle \varphi_*(p), A\partial_t n \rangle &\leq c(|A^{1/2}p|^2 + |p|^{q-2}|A^{1/2}p|^q) + H|A^{1/2}\partial_t n|^2 \\ &\leq c(|A^{1/2}p|^2 + \|p\|_q^{q-2}|A^{1/2}p|^q) + H|A^{1/2}\partial_t n|^2 \\ &\leq c\varpi^{-q/2} + H|A^{1/2}\partial_t n|^2 \end{aligned}$$

where Lemma 4.1 is used in the last inequality. Summarizing, for a fixed $t > s$, we arrive at the differential inequality

$$\frac{d}{d\tau}\Lambda_2(\tau) + H\Lambda_2(\tau) \leq c(1 + e^{qH\tau}) \leq c(1 + e^{qHt}), \quad \forall \tau \in [s, t].$$

An application of the Gronwall lemma then yields

$$(4.7) \quad \varpi \|An(t)\|^2 + \|A^{1/2}\partial_t n(t)\|^2 = \Lambda_2(t) \leq c(1 + e^{qHt}),$$

due to the fact that $\Lambda_2(s) = 0$. Combining (4.6)-(4.7), we finally arrive at

$$\|N_z(t, s)\|_{Y_t}^2 \leq c(1 + e^{qHt}) := h(t),$$

for all $t \in \mathbb{R}$, $s \leq t$ and $z \in \mathbb{A}(s)$, which is what we looked for. \square

We can now complete the proof of the main theorem.

Proof of Theorem 4.1. First of all, we remark again that each Y_t is compactly embedded in X_t and the injections $\mathcal{I}_t : Y_t \rightarrow X_t$ satisfy $\|\mathcal{I}_t\|_{\mathcal{L}(Y_t, X_t)} \leq \max\{1, \lambda_1^{-1}\}$. Finally, each Y_t is a reflexive Banach space, so that closed balls of Y_t are closed in X_t . These considerations ensure that we are in position to apply Corollaries 2.1 and 2.2.

Setting

$$P(t, s) = \bigcup_{z \in \mathbb{A}(s)} P_z(t, s), \quad N(t, s) = \bigcup_{z \in \mathbb{A}(s)} N_z(t, s),$$

we have $S(t, s)\mathbb{A}(s) \subset P(t, s) + N(t, s)$. Lemma 4.1 grants

$$\lim_{s \rightarrow -\infty} \|P(t, s)\|_{X_t} = \lim_{s \rightarrow -\infty} \sup_{z \in \mathbb{A}(s)} \|P_z(t, s)\|_{X_t} = 0,$$

(observe that convergence to zero in $\|\cdot\|_{X_t}$ is equivalent to convergence to zero of $\mathcal{E}_{X_t}(\cdot)$, as stated in Remark 3.2), while Lemma 4.2 establishes that

$$\sup_{s \leq t} \|N(t, s)\|_{Y_t}^2 \leq h(t).$$

Therefore, $N(t, s)$ is compact in X_t , for every $t \in \mathbb{R}$, $s \leq t$. Applying Corollary 2.1, we achieve the existence of the global attractor $\mathcal{A}(t) = \omega_{\mathbb{A}}(t)$. Finally, estimate (4.5) and Corollary 2.2 yields $\|\mathcal{A}(t)\|_{Y_t}^2 \leq h(t)$, and the uniqueness of \mathcal{A} in the sense of Remark 2.1. \square

5. THE PHYSICAL POTENTIALS.

In this short section we apply the general results obtained in Sect. 3 and 4 to the potentials discussed in the physics literature

$$V_{\pm}(y) = \frac{1}{2}y^2 \pm \frac{1}{4}y^4 + \frac{1}{6}y^6$$

(see also Remark 3.1). We also make some remarks concerning the forward asymptotic behavior (i.e. we fix an initial time $s \in \mathbb{R}$ and let $t \rightarrow +\infty$), for the whole class of potentials satisfying our assumptions (in particular, V_{\pm}).

5.1. Pullback exponential decay of (1.1) with V_{\pm} . Apart from the regularity result of Theorem 4.1, we do not dwell on the structure of the global pullback attractor for a general potential satisfying our assumptions (H0),(H1). However, for both potentials V_+, V_- described in Remark 3.1, the pullback attractor \mathcal{A} is reduced to zero, i.e. $\mathcal{A}(t) = \{0\}$ for every $t \in \mathbb{R}$.

Indeed, both V_- and V_+ comply with assumptions (H0)-(H1), with (in particular) $a_1 = 0$. As a result, we can take $c_0 = 0$ in (3.2), and (3.3) holds with $b_1 = 0$. We then read in Theorem 3.1 that estimate (3.5) holds with $K_1 = 0$ and can therefore be rewritten as

$$(5.1) \quad e^{-2Ht} |A^{1/2}u(t)|^2 + \|u(t)\|_{L^q}^q + |\partial_t u(t)|^2 \leq cR(t)e^{-\mu(t-s)}, \quad \forall t \geq s$$

whenever $z \in \mathcal{B}(s)$, with

$$R(t) = \sup_{s \leq t} \sup_{z \in \mathcal{B}(s)} \mathcal{E}_{X_s}(z) < \infty.$$

Hence, the rhs of (5.1), and thus the lhs, go to zero as $s \rightarrow -\infty$. We conclude that the potentials V_+, V_- (like any admissible potential fulfilling (H1) with $a_1 = 0$) produce (pullback) exponential decay of the solutions originating from pullback-bounded initial data. In other words, the family $\{\mathcal{A}(t) = \{0\}\}_{t \in \mathbb{R}}$ is the (unique, in the sense of Remark 2.1) time-dependent global attractor for the process generated by (P).

Remark 5.1. It is observed in the physical literature (see also the caption to Figure 1) that the term $\frac{y^6}{6}$ in the potentials V_{\pm} does not have a deep physical meaning. Accordingly, we consider the modified potentials

$$V_{\alpha\pm}(y) = \frac{1}{2}y^2 \pm \frac{1}{4}y^4 + \frac{\alpha}{6}y^6$$

for $\frac{1}{4} < \alpha < \frac{9}{20}$. The potentials $V_{\alpha-}$ still have a unique local and global minimum at 0, but fail to be convex, and henceforth the constant a_1 appearing in (H1) must be taken strictly positive. On the contrary, a_1 can be chosen to be zero for $V_{\alpha+}$. Therefore, we still have pullback exponential decay for $V_{\alpha+}$, whereas the result is not known for $V_{\alpha-}$; that is, in this case we cannot conclude that the pullback attractor is trivial.

The range $0 < \alpha < \frac{1}{4}$ for $V_{\alpha-}$, where two nontrivial (i.e. negative) global minima appear, is less relevant for the physical problem under consideration: however, assumptions (H0)-(H1) (with, necessarily, $a_1 > 0$) are still valid.

5.2. Forward convergence. We now consider the oscillon equation between times s and t , $s \in \mathbb{R}$ fixed, $t > s$ with the aim of letting $t \rightarrow +\infty$. We assume that the initial data $z = (u_0, v_0)$ satisfies $\mathcal{E}_{X_s}(z) \leq R$. We then infer from (3.5) that

$$(5.2) \quad e^{-2Ht} |A^{1/2}u(t)|^2 + \|u(t)\|_{L^q}^q + |\partial_t u(t)|^2 \leq K_0 R e^{-\mu(t-s)} + K_1 \quad \forall t \geq s,$$

where $0 < \mu < H$, K_0 and $K_1 = c(c_0 + b_1)$ have been computed in Theorem 3.1.

As mentioned above, in the case of the potentials V_- and V_+ , $K_1 = 0$. Consequently, we observe that, the solution $(u(t), \partial_t u(t)) = S(t, s)z$ decays exponentially to zero in the norm $L^q \times L^2$ as $t \rightarrow +\infty$ (forward convergence). Due to the fact that the bound on the norm of the initial data z depends on the initial time s , we cannot conclude that the above mentioned convergence is uniform in s . Therefore, we cannot conclude that $\{0\}$ is the (weak, i.e. $L^q \times L^2$) uniform attractor for $S(t, s)$ in the sense of Babin and Vishik

[2, 9]. Furthermore, since μ must be less than H in (3.5), nothing can be said about the behavior of $|A^{1/2}u(t)|$ as $t \rightarrow +\infty$ (while the pullback approach grants a bound at every fixed time t , see above). *Hence, this analysis leaves also open the question of the behavior of $|A^{1/2}u(t)|^2$ as $t \rightarrow +\infty$.*

With a further analysis, we can extend the result of decay of $\partial_t u$ described above to the whole class of potentials satisfying assumptions (H0)-(H1), though the rate of decay will be only polynomial in time, instead of exponential as in the case of potentials V_\pm . The proof of the following proposition relies on an adaptation of the argument in [3, Lemma 2.7].

Proposition 5.1. *Let $s \in \mathbb{R}$ be fixed, $z = (u_0, v_0) \in X$ such that $\mathcal{E}_{X_s}(z) \leq R$. We have the estimate*

$$(5.3) \quad \|\partial_t u(t)\|^2 \leq \mathcal{Q}_2(R) \frac{1}{1 + (t - s)},$$

for every $t \geq s$, where \mathcal{Q}_2 is specified below.

Proof. Setting

$$\mathcal{E}(t) = e^{-2Ht} |A^{1/2}u(t)|^2 + |\partial_t u(t)|^2, \quad \Phi(t) = \mathcal{E}(t) + 2\mathcal{V}(u(t)),$$

we rewrite (3.7) as

$$(5.4) \quad \frac{d}{dt} \Phi + 2H\mathcal{E} = 0.$$

Also note, as a consequence of (3.3), that

$$(5.5) \quad \Phi(t) \geq -2b_1, \quad \forall t \geq s.$$

A further consequence of (3.3) is that

$$(5.6) \quad \Phi(s) = e^{-2Hs} |A^{1/2}u_0|^2 + |v_0|^2 + 2\mathcal{V}(u_0) \leq c(1 + R + R^{2/q}) := \mathcal{Q}(R)$$

whenever $\mathcal{E}_{X_s}(z) \leq R$.

Let $\delta > 0$ be given and set

$$t_\delta = s + \frac{\mathcal{Q}(R) + 2b_1}{2H\delta}.$$

We will show that

$$(5.7) \quad \frac{d}{dt} \Phi(t) \geq -2H\delta$$

for every $t \geq t_\delta$. We first show that there exists $t_0 \in [s, t_\delta]$ such that (5.7) holds for $t = t_0$. Suppose it were not so, then $\frac{d}{dt} \Phi(t) < -2H\delta$ on $[s, t_\delta]$, which would imply, by integration,

$$\Phi(t_\delta) < \Phi(s) - 2H\delta(t_\delta - s) \leq \mathcal{Q}(R) - 2H\delta \frac{\mathcal{Q}(R) + 2b_1}{2H\delta} = -2b_1$$

which contradicts (5.5). Now, define

$$t^* = \sup \{ \tau \geq t_0 : (5.7) \text{ holds } \forall t \in [t_0, \tau] \}.$$

We show that $t^* = +\infty$. Indeed, if it were not so we could pick $t_n \downarrow t^*$, for which $\frac{d}{dt}\Phi(t_n) < -2H\delta$. Consequently, for any given $\varepsilon > 0$, we can find another sequence τ_n , with $t^* < \tau_n < t_n$, such that

$$\Phi(\tau_n) > \Phi(t^*) - \varepsilon, \quad \Phi(\tau_n) < \Phi(t_n).$$

This implies that $\Phi(t^*) < \Phi(t_n) + \varepsilon$, and henceforth, ε being arbitrary, $\Phi(t_n) - \Phi(t^*) \geq 0$. In turn, this leads to $\frac{d}{dt}\Phi(t^*) \geq 0$. By continuity of $\frac{d}{dt}\Phi$ (which is a consequence of (5.4) and the continuity properties of the solutions), $\frac{d}{dt}\Phi(t) \geq 0$ in a right neighborhood of t^* , which contradicts maximality.

Hence, (5.7) holds for all $t \geq t_\delta$: inserting this in (5.4) yields immediately

$$|\partial_t u(t)|^2 \leq \mathcal{E}(t) \leq \delta, \quad \forall t \geq t_\delta.$$

To get (5.3) for a given $t > s + 1$, it is sufficient to choose $\delta = \frac{\mathcal{Q}(R) + 2b_1}{2H(t-s)}$, so that $t_\delta = t$, and

$$|\partial_t u(t)|^2 \leq \frac{\mathcal{Q}(R) + 2b_1}{2H(t-s)}.$$

On the other hand, if $0 \leq t - s < 1$, integrating (5.4) gives

$$|\partial_t u(t)|^2 \leq \Phi(t) + 2b_1 \leq \Phi(s) + 2b_1 \leq \mathcal{Q}(R) + 2b_1.$$

Combining the last two inequalities yields (5.3) with $\mathcal{Q}_2(R) = (\mathcal{Q}(R) + 2b_1) \max\{1, H^{-1}\}$. \square

6. FINITE-DIMENSIONALITY OF THE GLOBAL ATTRACTOR.

We conclude the paper with a study of the fractal dimension of the pullback attractor \mathcal{A} of system (P) constructed above.

Fractal dimension. For a Banach space W , let B_W denote the closed unit ball of W . For $\varepsilon > 0$, we call ε -ball centered at $x \in W$ the set $B_W(\varepsilon, x) = x + \varepsilon B_W$.

If $K \subset W$ is compact, we use $\mathcal{N}_\varepsilon(K, W)$ to indicate the minimum number of ε -balls of W which cover K , and we define the *fractal dimension* of K as

$$\dim_W K = \limsup_{\varepsilon \rightarrow 0^+} \frac{\log \mathcal{N}_\varepsilon(K, W)}{\log \frac{1}{\varepsilon}}$$

For more details on the fractal dimension (also known as the Minkowski or *box-counting* dimension), we refer the reader to e.g. [22, 28] (see also [30]).

Remark 6.1. Directly from the definition, it follows that Banach space isomorphisms preserve fractal dimension: more precisely, if W, \widetilde{W} are two Banach spaces, $K \subset W$ is compact and $J : W \rightarrow \widetilde{W}$ is a Banach space isomorphism, then $\dim_{\widetilde{W}} J(K) = \dim_W K$.

The following abstract lemma (an adaptation of the generalized squeezing property method [14, 16] to our framework) is the main technical tool we need in order to establish a bound on the fractal dimension of $\mathcal{A}(t)$.

Lemma 6.1. *For $k \in \mathbb{N}$, let W_k, Z_k be two families of Banach spaces satisfying*

- (i) $Z_k \subseteq W_k$;
- (ii) *for each $\varepsilon > 0$, $\kappa_\varepsilon = \sup_{k \geq 0} \mathcal{N}_\varepsilon(B_{Z_k}(1, 0), W_k) < \infty$.*

Let $\mathcal{B} = \{\mathcal{B}_k \subset W_k\}_k$ be a family of sets with maps $U^k : \mathcal{B}_k \rightarrow \mathcal{B}_{k-1}$, $k \geq 1$, fulfilling

- (iii) \mathcal{B}_k is compact in W_k ;
- (iv) $\sup_{k \geq 0} \|\mathcal{B}_k\|_{W_k} = Q_1 < \infty$;
- (v) $U^k(\mathcal{B}_k) = \mathcal{B}_{k-1}$;
- (vi) there exists a decomposition $U^k(z) = P^k(z) + N^k(z)$ and constants $\varrho < \frac{1}{4}$, $Q_2 > 0$ such that

$$(6.1) \quad \|P^k(z^1) - P^k(z^2)\|_{W_{k-1}} \leq \varrho \|z^1 - z^2\|_{W_k},$$

and

$$(6.2) \quad \|N^k(z^1) - N^k(z^2)\|_{Z_{k-1}} \leq Q_2 \|z^1 - z^2\|_{W_k},$$

for every $z^1, z^2 \in \mathcal{B}_k$.

Then,

$$\dim_{W_0} \mathcal{B}_0 \leq \frac{\log_2 \kappa_{\varrho Q_2^{-1}}}{\log_2 \frac{1}{4\varrho}}.$$

Proof. Let $0 < \varepsilon < Q_2$ and $k \geq 1$ be fixed. By compactness, \mathcal{B}_k can be covered by a finite number $\eta_\varepsilon = \eta_\varepsilon(\mathcal{B}_k, W_k)$ of ε -balls $\{B_{W_k}(\varepsilon, z^i)\}_{i=1}^{\eta_\varepsilon}$ with $z^i \in \mathcal{B}_k$. Then, from (6.2), we learn that for any fixed $z \in \mathcal{B}_k$ there exists z^i such that

$$\|N^k(z) - N^k(z^i)\|_{Z_{k-1}} \leq \varepsilon Q_2 \implies N^k(z) \in B_{Z_{k-1}}(\varepsilon Q_2, N^k(z^i)),$$

so that $N^k(\mathcal{B}_k)$ is covered by the set of balls $\{B_{Z_{k-1}}(\varepsilon Q_2, N^k(z^i))\}_{i=1}^{\eta_\varepsilon}$. Now, we cover each ball in this set by a finite number of $\varrho\varepsilon$ -balls of W_{k-1} . The minimum number of such balls is given by

$$\mathcal{N}_{\varrho\varepsilon}(B_{Z_{k-1}}(\varepsilon Q_2, 0), W_{k-1}) = \mathcal{N}_{\varrho Q_2^{-1}}(B_{Z_{k-1}}(1, 0), W_{k-1}) = \kappa_{\varrho Q_2^{-1}} := \kappa.$$

Hence, there exists a collection $\{B_{W_{k-1}}(\varrho\varepsilon, y^{i,j})\}_{i,j=1}^{\eta_\varepsilon, \kappa}$, with $y^{i,j} \in W_{k-1}$, covering $N^k(\mathcal{B}_k)$. This means that, for any fixed $z \in B_{W_k}(\varepsilon, z^i)$, there exist $j, y^{i,j}$ such that

$$\|U^k(z) - (y^{i,j} + P^k(z^i))\|_{W_{k-1}} \leq \|N^k(z) - y^{i,j}\|_{W_{k-1}} + \|P^k(z) - P^k(z^i)\|_{W_{k-1}} \leq 2\varrho\varepsilon,$$

i.e. if $\beta := 4\varrho < 1$, $U^k(z) \in B_{W_{k-1}}(\frac{\beta\varepsilon}{2}, y^{i,j} + P^k(z^i))$, so that

$$U^k \mathcal{B}_k \subset \bigcup_{i=1}^{\eta_\varepsilon} \bigcup_{j=1}^{\kappa} B^{i,j}, \quad B^{i,j} = B_{W_{k-1}}\left(\frac{\beta\varepsilon}{2}, y^{i,j} + P^k(z^i)\right).$$

It might happen that some $y^{i,j} + P^k(z^i)$ does not belong to \mathcal{B}_{k-1} . In this case, choose $\tilde{y} \in \mathcal{B}_{k-1} \cap B^{i,j}$ and replace $B^{i,j}$ with $B_{W_{k-1}}(\beta\varepsilon, \tilde{y})$. Therefore, a system of $\kappa\eta_\varepsilon$ $\beta\varepsilon$ -balls of W_{k-1} is sufficient to cover \mathcal{B}_{k-1} , and we can estimate

$$(6.3) \quad \log_2 \mathcal{N}_{\beta\varepsilon}(\mathcal{B}_{k-1}, W_{k-1}) \leq \log_2 \kappa + \log_2 \mathcal{N}_\varepsilon(\mathcal{B}_k, W_k).$$

We then learn from (iv) that $\mathcal{N}_{Q_1}(\mathcal{B}_k, W_k) = 1$ for every $k \geq 0$, and, using (6.3) k times starting from \mathcal{B}_k , obtain that

$$\log_2 \mathcal{N}_{\beta^k Q_1}(\mathcal{B}_0, W_0) \leq k \log_2 \kappa + \log_2 \mathcal{N}_{Q_1}(\mathcal{B}_k, W_k) = k \log_2 \kappa.$$

Hence, if $\varepsilon > 0$ is arbitrary and k is chosen so that $\beta^k Q_1 \leq \varepsilon \leq \beta^{k-1} Q_1$, we have

$$\log_2 \mathcal{N}_\varepsilon(\mathcal{B}_0, W_0) \leq \log_2 \mathcal{N}_{\beta^k Q_1}(\mathcal{B}_0, W_0) \leq k \log_2 \kappa \leq \left(\log_2 \frac{1}{\beta}\right)^{-1} \log_2 \left(\frac{Q_1}{\beta\varepsilon}\right) \log_2 \kappa.$$

Dividing by $\log_2 \frac{1}{\varepsilon}$, rearranging and letting $\varepsilon \rightarrow 0$ yields the claim. \square

We are now ready to state and prove the main result of the section, that is, an upper bound on the fractal dimension of the sections $\mathcal{A}(t)$ of the global attractor \mathcal{A} constructed in Sect. 3.

Theorem 6.1. *Assume, in addition to (H0)-(H1), that $\varphi \in \mathcal{C}^2(\mathbb{R})$ and*

$$(6.4) \quad |\varphi''(y)| \leq c(1 + |y|^\vartheta), \quad \vartheta = \max\{q - 3, 0\}.$$

Then, the global attractor $\mathcal{A} = \{\mathcal{A}(t)\}_{t \in \mathbb{R}}$ has finite fractal dimension in X_t for every $t \in \mathbb{R}$, i.e.

$$\dim_{X_t} \mathcal{A}(t) \leq h_1(t), \quad \forall t \in \mathbb{R}.$$

Moreover, the positive increasing function h_1 depends only on the physical parameters of the problem and can be explicitly computed.

Proof of Theorem 6.1. To be in position to apply Lemma 6.1, we need to establish a suitable smoothing property for the process $S(\cdot, \cdot)$ restricted to the family \mathcal{A} . To do so, we need to separate the mean value of the solution $u = u(x, t)$ on $(0, 1)$. From now on, we write \hat{f} to indicate the mean value of $f \in L^1(0, 1)$ and use the notation $\tilde{f} = f - \hat{f}$.

Let $s \in \mathbb{R}$, and $z = (u_0, v_0) \in \mathcal{A}(s)$. We decompose $S(t, s)z$ as

$$S(t, s)z = (u(t), \partial_t u(t)) = \widehat{S}_z(t, s) + \widetilde{S}_z(t, s), \quad t \geq s,$$

where $\widehat{S}_z(t, s) = (\widehat{u}(t), \widehat{\partial_t u}(t))$, and $\widetilde{S}_z(t, s) = S(t, s)z - \widehat{S}_z(t, s)$.

Denote with \widetilde{L}^2 [resp. H_{per}^i , $i = 1, 2$] the subspace of functions of $L^2(0, 1)$ [resp. $H_{\text{per}}^i(0, 1)$] with zero mean value. We will look at the evolution of the zero mean part of the solution in the families of Banach spaces $(t \in \mathbb{R})$

$$\begin{aligned} \widetilde{X}_t &= \widetilde{H_{\text{per}}^1} \times \widetilde{L}^2, & \|(u, v)\|_{\widetilde{X}_t}^2 &= e^{-2Ht} |A^{1/2}u|^2 + |v|^2, \\ \widetilde{Y}_t &= \widetilde{H_{\text{per}}^2} \times \widetilde{H_{\text{per}}^1}, & \|(u, v)\|_{\widetilde{Y}_t}^2 &= e^{-2Ht} |Au|^2 + |A^{1/2}v|^2, \end{aligned}$$

Lemma 6.2. *Let $t_0 \in \mathbb{R}$, $t_\star > 0$ be fixed, $s \leq t_0 - t_\star$, $z^1, z^2 \in \mathcal{A}(s)$. We have the estimate*

$$(6.5) \quad \|\widehat{S}_{z^1}(s + t_\star, s) - \widehat{S}_{z^2}(s + t_\star, s)\|_{\mathbb{R}^2}^2 \leq ce^{C_0 t_\star} \|\hat{z}^1 - \hat{z}^2\|_{\mathbb{R}^2}^2,$$

where $C_0 > 0$ depends only on the physical parameters of the problem.

Proof. Fix $s \leq t_0 - t_\star$ and assume $t \in [s, s + t_\star]$. For $\iota = 1, 2$, let $z^\iota = (u_0^\iota, v_0^\iota) \in \mathcal{A}(s)$, and $S(t, s)z^\iota = (u^\iota(t), \partial_t u^\iota(t))$. We begin by observing that the difference $(m(t), m'(t)) = \widehat{S}_{z^1}(t, s) - \widehat{S}_{z^2}(t, s)$ fulfills the Cauchy problem on $(s, s + t_\star)$

$$m''(t) + Hm'(t) + g_1(t)m(t) = 0, \quad m(s) = \hat{u}_0^1 - \hat{u}_0^2, \quad m'(s) = \hat{v}_0^1 - \hat{v}_0^2,$$

with

$$g_1(t) = \int_0^1 \varphi'(\xi(x, t)) dx, \quad \min\{u^1(x, t), u^2(x, t)\} \leq \xi(x, t) \leq \max\{u^1(x, t), u^2(x, t)\}.$$

Now, set $\Lambda_m = (m')^2 + \frac{H^2}{4}m^2 + \frac{H}{2}mm'$. It is immediate to determine the differential inequality

$$\frac{d}{dt}\Lambda_m = -\frac{3}{2}H(m')^2 - 2g_1m(m' + \frac{H}{4}m) \leq c|g_1|(m^2 + (m')^2) \leq c(1 + R_{\mathbb{A}}^{\frac{q-2}{q}})\Lambda_m.$$

Here, using (H1) and (4.1), we have written

$$|g_1(t)| \leq \|\varphi'(\xi(t))\|_{L^{\frac{q}{q-2}}} \leq c(1 + \|u^1(t)\|_{L^q}^{q-2} + \|u^2(t)\|_{L^q}^{q-2}) \leq c(1 + R_{\mathbb{A}}^{\frac{q-2}{q}}).$$

An application of Gronwall's lemma on (s, t) then leads to

$$(6.6) \quad |m'(t)|^2 + |m(t)|^2 \leq c\Lambda_m(t) \leq ce^{C_0(t-s)}\Lambda_m(s) \leq ce^{C_0t_*}\|\hat{z}^1 - \hat{z}^2\|_{\mathbb{R}^2}^2,$$

for $C_0 = c(1 + R_{\mathbb{A}}^{\frac{q-2}{q}})$. Finally, writing (6.6) for $t = s + t_*$ gives (6.5). \square

Lemma 6.3. *Let $t_0 \in \mathbb{R}, t_* > 0$ be fixed, $s \leq t_0 - t_*$. There exists a decomposition*

$$\tilde{S}_z(t, s) = P(t, s)[z] + N(t, s)[z], \quad z \in \mathcal{A}(s),$$

satisfying, for every $z^1, z^2 \in \mathcal{A}(s)$,

$$(6.7) \quad \|P(s + t_*, s)[z^1] - P(s + t_*, s)[z^2]\|_{\tilde{X}_{s+t_*}}^2 \leq e^{-2Ht_*}\|\hat{z}^1 - \hat{z}^2\|_{\tilde{X}_s}^2$$

and

$$(6.8) \quad \|N(s + t_*, s)[z^1] - N(s + t_*, s)[z^2]\|_{\tilde{Y}_{s+t_*}}^2 \leq C_{t_0, t_*} [\|\hat{z}^1 - \hat{z}^2\|_{\mathbb{R}^2}^2 + \|\hat{z}^1 - \hat{z}^2\|_{\tilde{X}_s}^2]$$

where $C_{t_0, t_} > 0$ depends only on t_0, t_* and on the physical parameters of the problem.*

Proof. We will again use the shorthand $\varpi(t) = e^{-2Ht}$, and repeatedly exploit the inequality

$$(6.9) \quad \|u\|_{L^\infty}^2 \leq c|u|A^{1/2}u \leq cR_{\mathbb{A}}\varpi^{-1/2},$$

which holds for every trajectory $u = u(\cdot)$ on the attractor $\mathcal{A}(\cdot)$. Here, $R_{\mathbb{A}}$ is the radius of the absorbing set specified in (3.17).

Throughout, assume $t \in [s, s + t_*]$. We decompose

$$\tilde{S}_z(t, s) = P(t, s)[z] + N(t, s)[z] = (p(t), \partial_t p(t)) + (n(t), \partial_t n(t))$$

where

$$(6.10) \quad \begin{cases} \partial_{tt}p(t) + H\partial_t p(t) + e^{-2Ht}Ap(t) = 0, \\ p(s) = \tilde{u}_0, \partial_t p(s) = \tilde{v}_0, \end{cases}$$

and

$$(6.11) \quad \begin{cases} \partial_{tt}n(t) + H\partial_t n(t) + e^{-2Ht}An(t) = \widehat{\varphi(u(t))} - \varphi(u(t)), \\ n(s) = 0, \partial_t n(s) = 0. \end{cases}$$

The usual multiplication of (6.10) by $\partial_t p$ and Gronwall's lemma on (s, t) give

$$(6.12) \quad \|P(t, s)[z]\|_{\tilde{X}_t}^2 \leq e^{-2H(t-s)}\|\tilde{z}\|_{\tilde{X}_s}^2.$$

Using that $z \mapsto P(t, s)[z]$ is linear, (6.7) follows from (6.12) written for $t = s + t_*, z = z^1 - z^2$.

We turn to the difference $(\bar{n}(t), \partial_t \bar{n}(t)) = N(t, s)[z^1] - N(t, s)[z^2]$, which is a solution to

$$(6.13) \quad \begin{cases} \partial_{tt} \bar{n}(t) + H \partial_t \bar{n}(t) + e^{-2Ht} A \bar{n}(t) = g_2(t) + g_3(t), \\ \bar{n}(s) = 0, \partial_t \bar{n}(s) = 0. \end{cases}$$

with (here, ξ is as above)

$$g_2 = \widehat{\varphi(u^1)} - \widehat{\varphi(u^2)} = \int_0^1 \varphi'(\xi(x, \cdot))(u^1(x, \cdot) - u^2(x, \cdot)) dx, \quad g_3 = -(\varphi(u^1) - \varphi(u^2)).$$

We first multiply (6.13) by $\partial_t \bar{n}$ and obtain the differential inequality

$$(6.14) \quad \frac{d}{dt} \|(\bar{n}, \partial_t \bar{n})\|_{\tilde{X}_t}^2 + 2H |\partial_t \bar{n}|^2 \leq 2 \langle g_2 + g_3, \partial_t \bar{n} \rangle \leq c[|g_2|^2 + |g_3|^2] + 2H |\partial_t \bar{n}|^2.$$

The nonlinear terms are estimated as

$$(6.15) \quad |g_2|^2 \leq |\varphi'(\xi)|^2 |u^1 - u^2|^2 \leq (1 + \|u^1\|_{L^\infty}^{2(q-2)} + \|u^2\|_{L^\infty}^{2(q-2)}) |u^1 - u^2|^2,$$

and

$$(6.16) \quad \begin{aligned} |g_3|^2 &= |\varphi'(\xi)(u^1 - u^2)|^2 \leq \|\varphi'(\xi)\|_{L^\infty}^2 |u^1 - u^2|^2 \\ &\leq c(1 + \|u^1\|_{L^\infty}^{2(q-2)} + \|u^2\|_{L^\infty}^{2(q-2)}) |u^1 - u^2|^2. \end{aligned}$$

Therefore, writing $u^1 - u^2 = m + \bar{p} + \bar{n}$, where $(\bar{p}(t), \partial_t \bar{p}(t)) = P(t, s)[z^1] - P(t, s)[z^2]$, and exploiting (6.9) to control $\|u^i\|_{L^\infty}$, we collect the above estimates into

$$\begin{aligned} |g_2|^2 + |g_3|^2 &\leq c(1 + R_{\mathbb{A}}^{q-2} \varpi^{-q/2+1}) (m^2 + |\bar{p}|^2 + |\bar{n}|^2) \\ &\leq c(1 + R_{\mathbb{A}}^{q-2} \varpi^{-q/2+1}) (m^2 + |A^{1/2} \bar{p}|^2 + |A^{1/2} \bar{n}|^2) \\ &\leq c(1 + R_{\mathbb{A}}^{q-2} \varpi^{-q/2}) [m^2 + \|(\bar{p}, \partial_t \bar{p})\|_{\tilde{X}_t}^2 + \|(\bar{n}, \partial_t \bar{n})\|_{\tilde{X}_t}^2] \end{aligned}$$

Therefore, setting $C_1(t) = (1 + R_{\mathbb{A}}^{q-2} \varpi^{-q/2}(t))$, and observing that $C_1(t) \leq C_1(t_0)$, (6.14) turns into

$$(6.17) \quad \begin{aligned} \frac{d}{dt} \|(\bar{n}, \partial_t \bar{n})\|_{\tilde{X}_t}^2 &\leq C_1(t_0) [m^2 + \|(\bar{p}, \partial_t \bar{p})\|_{\tilde{X}_t}^2 + \|(\bar{n}, \partial_t \bar{n})\|_{\tilde{X}_t}^2] \\ &\leq cC_1(t_0) e^{C_0 t_*} [\|\hat{z}^1 - \hat{z}^2\|_{\mathbb{R}^2}^2 + \|\tilde{z}^1 - \tilde{z}^2\|_{\tilde{X}_s}^2 + \|(\bar{n}, \partial_t \bar{n})\|_{\tilde{X}_t}^2], \end{aligned}$$

where we used (6.6) and (6.12) in the last passage. We apply Gronwall's lemma on (s, t) , and obtain the intermediate estimate

$$(6.18) \quad \|(\bar{n}(t), \partial_t \bar{n}(t))\|_{\tilde{X}_t}^2 \leq e^{t_* C_2(t_0, t_*)} [\|\hat{z}^1 - \hat{z}^2\|_{\mathbb{R}^2}^2 + \|\tilde{z}^1 - \tilde{z}^2\|_{\tilde{X}_s}^2], \quad t \in [s, s + t_*],$$

with $C_2(t_0, t) = \exp(cC_1(t_0)e^{C_0 t_*})$. Now, a further multiplication of (6.13) by $A \partial_t \bar{n}$ yields the differential inequality

$$(6.19) \quad \frac{d}{dt} \|(\bar{n}, \partial_t \bar{n})\|_{\tilde{Y}_t}^2 + 2H |A^{1/2} \partial_t \bar{n}|^2 \leq 2 \langle g_2 + g_3, A \partial_t \bar{n} \rangle \leq c |A^{1/2} g_3|^2 + 2H |A^{1/2} \partial_t \bar{n}|^2,$$

due to the fact that g_2 is independent of x . For the remaining nonlinear term, we use the mean value theorem ($\varphi \in \mathcal{C}^2(\mathbb{R})$) and write

$$\partial_x g_3 = \varphi'(u^1) \partial_x u^1 - \varphi'(u^2) \partial_x u^2 = \varphi'(u^1) \partial_x (u^1 - u^2) + \varphi''(\eta) (u^1 - u^2) \partial_x u^2,$$

with $\eta = \eta(x, t)$ between $u^1(x, t)$ and $u^2(x, t)$. Thanks to (6.9), we obtain the controls

$$\begin{aligned} |\varphi'(u^1)\partial_x(u^1 - u^2)|^2 &\leq c(1 + \|u^1\|_{L^\infty}^{2(q-2)}) [|A^{1/2}\bar{p}|^2 + |A^{1/2}\bar{n}|^2] \\ &\leq c(1 + R_{\mathbb{A}}^{q-2}\varpi^{-q/2+1}) [|A^{1/2}\bar{p}|^2 + |A^{1/2}\bar{n}|^2] \end{aligned}$$

and, recalling (6.4),

$$\begin{aligned} |\varphi''(\eta)(u^1 - u^2)\partial_x u^2|^2 &\leq c(1 + \|u_1\|_{L^\infty}^{2\vartheta} + \|u_2\|_{L^\infty}^{2\vartheta}) |u^1 - u^2|^2 |A^{1/2}u^2|^2 \\ &\leq c(1 + R_{\mathbb{A}}^{1+\vartheta}\varpi^{-1-\frac{\vartheta}{2}}) (m^2 + |A^{1/2}\bar{p}|^2 + |A^{1/2}\bar{n}|^2). \end{aligned}$$

Being $1 + \vartheta \leq q - 2$, and $\tilde{\vartheta} := 1 + \frac{\vartheta}{2} > \frac{q}{2} - 1$, we summarize the above bounds into

$$|A^{1/2}g_3|^2 \leq c|\partial_x g_3|^2 \leq c(1 + R_{\mathbb{A}}^{q-2}\varpi^{-1-\tilde{\vartheta}}) [\|(\bar{p}, \partial_t \bar{p})\|_{\tilde{X}_t}^2 + \|(\bar{n}, \partial_t \bar{n})\|_{\tilde{X}_t}^2 + m^2].$$

Setting $C_3(t) = c(1 + R_{\mathbb{A}}^{q-2}\varpi^{-1-\tilde{\vartheta}})$, and again observing that $C_3(t) \leq C_3(t_0)$, (6.19) turns into

$$\begin{aligned} (6.20) \quad \frac{d}{dt} \|(\bar{n}, \partial_t \bar{n})\|_{\tilde{Y}_t}^2 &\leq C_3(t_0) [\|(\bar{p}, \partial_t \bar{p})\|_{\tilde{X}_t}^2 + \|(\bar{n}, \partial_t \bar{n})\|_{\tilde{X}_t}^2 + m^2] \\ &\leq C_3(t_0) e^{C_2(t_0, t_\star)t_\star} [\|\tilde{z}^1 - \tilde{z}^2\|_{\tilde{X}_s}^2 + \|\hat{z}^1 - \hat{z}^2\|_{\mathbb{R}^2}^2]. \end{aligned}$$

Here we used again (6.5), (6.7) and (6.18) in the last passage. Finally, we integrate on $(s, s + t_\star)$. Being $\bar{n}(s) = 0, \partial_t \bar{n}(s) = 0$, we end up with

$$\|(\bar{n}(s + t_\star), \partial_t \bar{n}(s + t_\star))\|_{\tilde{Y}_{s+t_\star}}^2 \leq C_{t_0, t_\star} [\|\tilde{z}^1 - \tilde{z}^2\|_{\tilde{X}_s}^2 + \|\hat{z}^1 - \hat{z}^2\|_{\mathbb{R}^2}^2],$$

which is (6.8), with $C_{t_0, t_\star} = t_\star C_3(t_0) e^{C_2(t_0, t_\star)t_\star}$. \square

We are ready to complete the proof of Theorem 6.1. To begin with, we identify each $z \in \mathcal{A}(t)$ with the pair $(\hat{z}, \tilde{z}) \in \mathbb{R}^2 \times \tilde{X}_t$. We then define the family

$$\mathcal{A}_\star = \left\{ \mathcal{A}_\star(t) = \{(\hat{z}, \tilde{z}) : z \in \mathcal{A}(t)\}, t \in \mathbb{R} \right\}, \quad \mathcal{A}_\star(t) \subset \mathbb{R}^2 \times \tilde{X}_t.$$

From the properties of \mathcal{A} (see Theorem 4.1), each $\mathcal{A}_\star(t)$ is bounded in $\mathbb{R}^2 \times \tilde{Y}_t$ and hence compact in $\mathbb{R}^2 \times \tilde{X}_t$. The bound (4.1) also guarantees that $\|\mathcal{A}_\star(t)\|_{\mathbb{R}^2 \times \tilde{X}_t} \leq Q_1$, for some positive Q_1 depending only on $R_{\mathbb{A}}$.

Now, fix $t_0 \in \mathbb{R}$ and set $t_\star = 3H^{-1} \log 2$. Referring to Lemma 6.1, for $k \in \mathbb{N} \cup \{0\}$, set

$$\begin{aligned} W_k &= \mathbb{R}^2 \times \tilde{X}_{t_0 - kt_\star}, & \|(\hat{z}, \tilde{z})\|_{W_k}^2 &= \|\hat{z}\|_{\mathbb{R}^2}^2 + \|\tilde{z}\|_{\tilde{X}_{t_0 - kt_\star}}^2, \\ Z_k &= \mathbb{R}^2 \times \tilde{Y}_{t_0 - kt_\star}, & \|(\hat{z}, \tilde{z})\|_{Z_k}^2 &= \|\hat{z}\|_{\mathbb{R}^2}^2 + \|\tilde{z}\|_{\tilde{Y}_{t_0 - kt_\star}}^2. \end{aligned}$$

We easily have $Z_k \subseteq W_k$ for each k , so that assumption (i) is verified. Then, we point out that the linear isomorphism

$$W_k \ni z = (\hat{z}, (\tilde{u}, \tilde{v})) \mapsto J_k(z) = (\hat{z}, (e^{-kt_\star H} \tilde{u}, \tilde{v})) \in W_0$$

satisfies $J_k(B_{W_k}) = B_{W_0}, J_k(B_{Z_k}) = B_{Z_0}$. Hence, for $\varepsilon > 0$,

$$\mathcal{N}_\varepsilon(B_{Z_k}, W_k) = N_\varepsilon(B_{Z_0}, W_0) = \kappa_\varepsilon < \infty,$$

so that (ii) is satisfied as well. Finally, we consider the sets $\mathcal{B}_k = \mathcal{A}_\star(t_0 - kt_\star)$, for which Properties (iii)-(iv) have been already verified above. For $k \geq 1$, define the maps $U^k : \mathcal{B}_k \rightarrow \mathcal{B}_{k-1}$,

$$z = (\hat{z}, \tilde{z}) \mapsto U^k((\hat{z}, \tilde{z})) = (\hat{S}_z(t_0 - (k-1)t_\star, t_0 - kt_\star), \tilde{S}_z(t_0 - (k-1)t_\star, t_0 - kt_\star)),$$

Due to the invariance of \mathcal{A} , the U^k are well defined and $U^k(\mathcal{B}_k) = \mathcal{B}_{k-1}$, which is (v). Regarding (vi), referring to Lemma 6.3 we write $U^k(z) = P^k(z) + N^k(z)$, with

$$\begin{aligned} P^k(z) &= (0, P(t_0 - (k-1)t_\star, t_0 - kt_\star)[z]), \\ N^k(z) &= (\hat{S}_z(t_0 - (k-1)t_\star, t_0 - kt_\star), N(t_0 - (k-1)t_\star, t_0 - kt_\star)[z]). \end{aligned}$$

Writing (6.7) for $s = t_0 - kt_\star$, we learn that

$$\|P^k(z^1) - P^k(z^2)\|_{W^{k-1}} \leq e^{-Ht_\star} \|z^1 - z^2\|_{W_k} = \frac{1}{8} \|z^1 - z^2\|_{W_k}, \quad \forall z^1, z^2 \in \mathcal{B}_k,$$

so that (6.1) is satisfied with $\varrho = \frac{1}{8}$. Moreover, collecting (6.5) and (6.8) written for $s = t_0 - kt_\star$, we have

$$\|N^k(z^1) - N^k(z^2)\|_{Z^{k-1}} \leq Q_{t_0} \|z^1 - z^2\|_{W_k} \quad \forall z^1, z^2 \in \mathcal{B}_k,$$

with $Q_{t_0} = (ce^{C_0 t_\star} + C_{t_0, t_\star})^{1/2}$, which is (6.2). Therefore, Lemma 6.1 applies, yielding

$$(6.21) \quad \dim_{\mathbb{R}^2 \times \tilde{X}_{t_0}} \mathcal{A}_\star(t_0) = \dim_{W_0} \mathcal{B}_0 \leq \log_2 \kappa_{(4Q_{t_0})^{-1}}.$$

To complete the proof, it is enough to recall that, for fixed t , $\mathbb{R}^2 \times \tilde{X}_t$ and X_t are isomorphic as Banach spaces through the map

$$\mathbb{R}^2 \times \tilde{X}_t \ni (\hat{z}, \tilde{z}) \mapsto J_t(\hat{z}, \tilde{z}) = \hat{z} + \tilde{z} \in X_t.$$

Indeed, J_t is clearly bijective and, if $(\hat{z}, \tilde{z}) = ((\hat{u}, \hat{v}), (\tilde{u}, \tilde{v}))$,

$$\begin{aligned} \|J_t(\hat{z}, \tilde{z})\|_{X_t}^2 &\leq 3 \left[\|\hat{u} + \tilde{u}\|_{L^q}^2 + e^{-2Ht} |A^{\frac{1}{2}} \tilde{u}|^2 + |\hat{v} + \tilde{v}|^2 \right] \leq 3 \left[\|\hat{z}\|_{\mathbb{R}^2}^2 + \|\tilde{u}\|_{L^\infty}^2 + \|\tilde{z}\|_{\tilde{X}_t}^2 \right] \\ &\leq c \left[\|\hat{z}\|_{\mathbb{R}^2}^2 + \lambda_1^{-1} |A^{1/2} \tilde{u}|^2 + \|\tilde{z}\|_{\tilde{X}_t}^2 \right] \leq c(1 + \lambda_1^{-1} e^{2Ht}) \|(\hat{z}, \tilde{z})\|_{\mathbb{R}^2 \times \tilde{X}_t}^2. \end{aligned}$$

Since $J_{t_0}(\mathcal{A}_\star(t_0)) = \mathcal{A}(t_0)$ and Banach space isomorphisms preserve fractal dimension, (6.21) implies

$$\dim_{X_{t_0}} \mathcal{A}(t_0) = \dim_{\mathbb{R}^2 \times \tilde{X}_{t_0}} \mathcal{A}_\star(t_0) \leq \log_2 \kappa_{(4Q_{t_0})^{-1}},$$

as well, which is the statement of Theorem 6.1, with $h_1(t_0) = \log_2 \kappa_{(4Q_{t_0})^{-1}}$.

Acknowledgments. This work was partially supported by the National Science Foundation under the grants NSF-DMS-0604235, NSF-DMS-0906440, and by the Research Fund of Indiana University.

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